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In the Spotlight

Redundancy for Safety

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by Vito Faraci Jr.

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Safety is difficult to measure quantitatively. Traditionally, safety is measured by counting and keeping records of accidents and injury rates. But such measurements are "after the fact." While they do provide important information, "before the fact" measurements would encourage actions to prevent or at least mitigate the undesired event. Thanks to the relatively new science called "reliability," pertinent "before the fact" probability of success and/or failure measurements can be made.

Although safety and reliability are not the same, often (but not always) increasing reliability has the effect of increasing safety. A common example is the single-engine airplane. An engine failing during flight is a very unsafe situation, and could be life threatening. As a private pilot, this example stands out vividly in my mind. No engine, even the most reliable, has a 100 percent probability of success (P_S) for, let's say, a three-hour flight. In other words, there is no 100 percent guarantee that an engine will run smoothly without failure during an entire three-hour interval. A major reason for multi-engine airplanes is the increase in safety. Twin-engine airplanes can and should be designed to fly on one engine in the event that the other engine fails. This capability adds not only reliability, but, more importantly, safety. Similarly, any three- or four-engine airplane should be designed to fly with one or more engines failed. With this capability, multi-engine airplanes can be thought of as systems with engines as "active redundant" components.

Now some logical questions will arise from several groups, including airplane manufacturers, the FAA, NASA, safety engineers, reliability engineers, pilots, airplane passengers and others. For example, what is the amount of safety increase with the addition of one or more engines? Is a twin-engine airplane twice as safe as a single-engine airplane?

Objective

The objective of this paper is not to try to quantify safety, but to show what is involved in quantifying the probability of success of systems utilizing redundancy, which, in many cases improves probability of success, which ultimately results in increased safety. In other words, to show how to calculate reliability of systems that utilize "redundant" components in parallel and, at the same time, illustrate the effect (probability of success increase) with each addition of a redundant component. It should be noted that reliability and failure rate of parallel configurations using unequal failure rate components can certainly be calculated, as shown in Figure 1. However, this paper will concentrate only on configurations (systems) using redundant components of equal failure rate.

Definitions

In the world of reliability, two items are said to be in "parallel" if system success means both components are operating, or one component is operating. The two diagrams in Figure 1 represent reliability block diagrams, both showing components in parallel. Systems can be comprised of three or more components in parallel. In those cases, system success must be clearly defined as to the minimum number of components that must operate for system success. For example, a three-engine airplane may be designed to fly with two engines operating, thus allowing for one engine to fail. On the other hand, a three-engine airplane may be designed to fly with only one engine operating, thus allowing two engines to fail.

Components in parallel are usually identical and, therefore, will have the same failure rate. However, this is not always the case. The configurations in Figure 1 both show two components in parallel. However, one shows a system with its component failure rates equal, and the other shows a system with unequal failure rates.

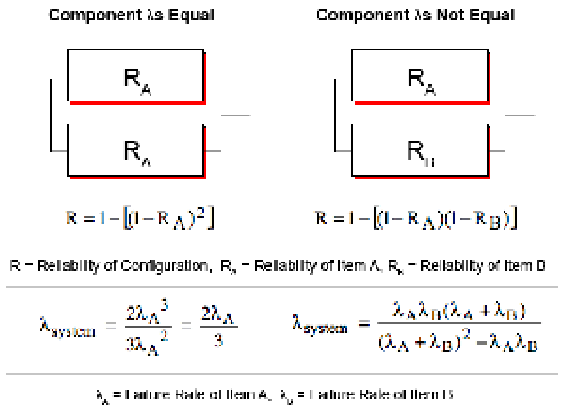

Figure 1 — A Reliability Block Diagram.

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Definitions (continued)

"m of n": m or more functional items required for system success where $1 \le m \le n$. In other words, at least m items are required to be functional (out of n) for system success.

Examples:

"3 of 3": Given a three-engine airplane, the airplane requires all three engines to fly. Note no failures allowed in this case.
"2 of 3": Given a three-engine airplane, the airplane requires two or more engines to fly. Note one failure allowed.
"1 of 3": Given a three-engine airplane, the airplane requires one or more engines to fly. Note two failures allowed.

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Probability of Success (P_S) and Mean Time to Failure (MTTF) Lookup Table

Let λ = failure rate of each item, t = time duration of a mission, $p=e^{-\lambda\tau}$ = probability of success of each item, then $q=1-e^{-\lambda\tau}$ = probability of failure of each item. The Lookup Table shown as Table 1 is constructed to show the system probability of success (P_S), system mean time to fail (MTTF), and system failure rate (FR = 1/MTTF) for any "m of n" configuration where $1 \le m \le n$.

Note: For a mathematician, the MTTF column of the table reveals something quite elegant.

Table 1 — P_S and MTTF Lookup Table.

m of n n = 2	System Probability of Success (P_s)	System (MTTF)	FR
2 of 2	$e^{-2\lambda t}$	$\frac{1}{\lambda}\left(\frac{1}{2}\right)$	2λ
1 of 2	$e^{-2\lambda t}+2(e^{-\lambda t}-e^{-2\lambda t})$	$\frac{1}{\lambda}\left(\frac{1}{2}+1\right)=\frac{3}{2\lambda}$	$\frac{2\lambda}{3}$
n = 3			
3 of 3	$e^{-3\lambda t}$	$\frac{1}{\lambda}\left(\frac{1}{3}\right)$	3λ
2 of 3	$e^{-3\lambda t}+3e^{-2\lambda t}(1-e^{-\lambda t})$	$\frac{1}{\lambda}\left(\frac{1}{3}+\frac{1}{2}\right)=\frac{5}{6\lambda}$	$\frac{6\lambda}{5}$
1 of 3	$e^{-3\lambda t}+3e^{-2\lambda t}(1-e^{-\lambda t})+3e^{-\lambda t}(1-e^{-\lambda t})^2$	$\frac{1}{\lambda}\left(\frac{1}{3}+\frac{1}{2}+1\right)=\frac{11}{6\lambda}$	$\frac{6\lambda}{11}$
n = 4			
4 of 4	$e^{-4\lambda t}$	$\frac{1}{\lambda}\left(\frac{1}{4}\right)$	4λ
3 of 4	$e^{-4\lambda t}+4e^{-3\lambda t}(1-e^{-\lambda t})$	$\frac{1}{\lambda}\left(\frac{1}{4}+\frac{1}{3}\right)=\frac{7}{12\lambda}$	$\frac{12\lambda}{7}$
2 of 4	$e^{-4\lambda t}+4e^{-3\lambda t}(1-e^{-\lambda t})+6e^{-2\lambda t}(1-e^{-\lambda t})^2$	$\frac{1}{\lambda}\left(\frac{1}{4}+\frac{1}{3}+\frac{1}{2}\right)=\frac{13}{12\lambda}$	$\frac{12\lambda}{13}$
1 of 4	$e^{-4\lambda t}+4e^{-3\lambda t}(1-e^{-\lambda t})+6e^{-2\lambda t}(1-e^{-\lambda t})^2$ $+4e^{-\lambda t}(1-e^{-\lambda t})^3$	$\frac{1}{\lambda}\left(\frac{1}{4}+\frac{1}{3}+\frac{1}{2}+1\right)=\frac{25}{12\lambda}$	$\frac{12\lambda}{25}$
n = 5			
5 of 5	$e^{-5\lambda t}$	$\frac{1}{\lambda}\left(\frac{1}{5}\right)$	5λ
4 of 5	$e^{-5\lambda t}+5e^{-4\lambda t}(1-e^{-\lambda t})$	$\frac{1}{\lambda}\left(\frac{1}{5}+\frac{1}{4}\right)=\frac{9}{20\lambda}$	$\frac{20\lambda}{9}$
3 of 5	$e^{-5\lambda t}+5e^{-4\lambda t}(1-e^{-\lambda t})+10e^{-3\lambda t}(1-e^{-\lambda t})^2$	$\frac{1}{\lambda}\left(\frac{1}{5}+\frac{1}{4}+\frac{1}{3}\right)=\frac{47}{60\lambda}$	$\frac{60\lambda}{47}$
2 of 5	$e^{-5\lambda t}+5e^{-4\lambda t}(1-e^{-\lambda t})+10e^{-3\lambda t}(1-e^{-\lambda t})^2$ $+10e^{-2\lambda t}(1-e^{-\lambda t})^3$	$\frac{1}{\lambda}\left(\frac{1}{5}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2}\right)=\frac{77}{60\lambda}$	$\frac{60\lambda}{77}$
1 of 5	$e^{-5\lambda t}+5e^{-4\lambda t}(1-e^{-\lambda t})+10e^{-3\lambda t}(1-e^{-\lambda t})^2$ $+10e^{-2\lambda t}(1-e^{-\lambda t})^3+5e^{-\lambda t}(1-e^{-\lambda t})^4$	$\frac{1}{\lambda}\left(\frac{1}{5}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2}+1\right)=\frac{137}{60\lambda}$	$\frac{60\lambda}{137}$
m of n	$1 < m < n$		
P_s	$\sum_{k=0}^{n-m}\binom{n}{k}e^{-(n-k)\lambda t}(1-e^{-\lambda t})^k$		
MTTF	$\frac{1}{\lambda}\left(\frac{1}{n}+\frac{1}{(n-1)}+\frac{1}{(n-2)}+\cdots+\frac{1}{(m+1)}+\frac{1}{m}\right)$		

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What follows next are the details of how the Lookup Table was constructed.

Note: The theorems and examples section that follows this article provides additional (and somewhat exhaustive, but necessary) mathematical detail to support what is presented.

Calculating Probability of Success and MTTF of "m of n" Parallel Configurations
(all items with equal λ)

Let $p = e^{-\lambda t}$ = probability of success of one item, and then $q = 1 - e^{-\lambda t}$ = probability of failure
Note: $p + q = 1$

Recall the Binomial Expansion $(p+q)^n = \binom{n}{0}p^n + \binom{n}{1}p^{n-1}q + \binom{n}{2}p^{n-2}q^2 + \cdots + \binom{n}{n}q^n$

where $\binom{n}{r} = nCr = \dfrac{n!}{(n-r)! \cdot r!}$

The n+1 terms of the expansion reveal the probabilities of exactly 0, 1, 2 ··· n failures as follows:

$$p + q = 1 \Rightarrow (p+q)^n = 1 = \binom{n}{0}p^n + \binom{n}{1}p^{n-1}q + \binom{n}{2}p^{n-2}q^2 + \cdots + \binom{n}{n}q^n \qquad (1)$$

P(exactly 0 failures) → $\binom{n}{0}p^n$
P(exactly 1 failure) → $\binom{n}{1}p^{n-1}q$
P(exactly 2 failures) → $\binom{n}{2}p^{n-2}q^2$
P(exactly n failures) → $\binom{n}{n}q^n$

By selecting correct terms of the above expansion, the probability of success of any "m of n" configuration can be calculated.

Case n = 2 (probability of success)

From equation (1) $(p+q)^2 = 1 = p^2 + 2pq + q^2 = e^{-2\lambda t} + 2e^{-\lambda t}(1 - e^{-\lambda t}) + (1 - e^{-\lambda t})^2 \Rightarrow$

$P(0\text{ fail}) = e^{-2\lambda t}$, $P(1\text{ fail}) = 2(e^{-\lambda t} - e^{-2\lambda t})$, $P(2\text{ fail}) = 1 - 2e^{-\lambda t} + e^{-2\lambda t} \Rightarrow$

$P(2\text{ of }2) = P(0\text{ fail}) = e^{-2\lambda t}$

$P(1\text{ of }2) = P(0\text{ fail}) + P(1\text{ fail}) = e^{-2\lambda t} + 2(e^{-\lambda t} - e^{-2\lambda t})$

Case n = 2 (MTTF)

Recall from probability theory $MTTF = \int_0^\infty P_s \, dt$ where P_S = probability of success =>

$$MTTF\ (2\text{ of }2) = \int_0^\infty P(2\text{ of }2)\, dt = \int_0^\infty e^{-2\lambda t}\, dt = \frac{1}{2\lambda}$$

$$MTTF\ (1\text{ of }2) = \int_0^\infty P(1\text{ of }2)\, dt = \int_0^\infty e^{-2\lambda t} dt + 2\int_0^\infty (e^{-\lambda t} - e^{-2\lambda t})\, dt = \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{3}{2\lambda}$$

Case n = 3 (probability of success)

$$\text{From equation (1)} \quad (p+q)^3 = 1 = p^3 + 3p^2q + 3pq^2 + q^3$$

$$= e^{-3\lambda t} + 3e^{-2\lambda t}(1-e^{-\lambda t}) + 3e^{-\lambda t}(1-e^{-\lambda t})^2 + (1-e^{-\lambda t})^3 \Rightarrow$$

$$P(0\ \text{fail}) = e^{-3\lambda t},\ P(1\ \text{fail}) = 3e^{-2\lambda t}(1-e^{-\lambda t}),\ P(2\ \text{fail}) = 3e^{-\lambda t}(1-e^{-\lambda t})^2,\ P(3\ \text{fail}) = (1-e^{-\lambda t})^3 \Rightarrow$$

$$P(3\ \text{of}\ 3) = P(0\ \text{fail}) = e^{-3\lambda t}$$

$$P(2\ \text{of}\ 3) = P(0\ \text{fail}) + P(1\ \text{fail}) = e^{-3\lambda t} + 3e^{-2\lambda t}(1-e^{-\lambda t})$$

$$P(1\ \text{of}\ 3) = P(0\ \text{fail}) + P(1\ \text{fail}) + P(2\ \text{fail}) = e^{-3\lambda t} + 3e^{-2\lambda t}(1-e^{-\lambda t}) + 3e^{-\lambda t}(1-e^{-\lambda t})^2$$

Case n = 3 (MTTF)

$$\text{MTTF}(3\ \text{of}\ 3) = \int_0^\infty P(3\ \text{of}\ 3)\,dt = \int_0^\infty e^{-3\lambda t}\,dt = \frac{1}{3\lambda}$$

$$\text{MTTF}(2\ \text{of}\ 3) = \int_0^\infty P(2\ \text{of}\ 3)\,dt = \int_0^\infty e^{-3\lambda t}\,dt + 3\int_0^\infty e^{-2\lambda t}(1-e^{-\lambda t})\,dt = \frac{1}{3\lambda} + \frac{1}{2\lambda} = \frac{5}{6\lambda}$$

$$\text{MTTF}(1\ \text{of}\ 3) = \int_0^\infty P(1\ \text{of}\ 3)\,dt = \int_0^\infty e^{-3\lambda t}\,dt + 3\int_0^\infty e^{-2\lambda t}(1-e^{-\lambda t})\,dt + 3\int_0^\infty e^{-\lambda t}(1-e^{-\lambda t})^2\,dt = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{11}{6\lambda}$$

Case n = 4 (probability of success)

$$\text{From equation (1)} \quad (p+q)^4 = 1 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$$

$$= e^{-4\lambda t} + 4e^{-3\lambda t}(1-e^{-\lambda t}) + 6e^{-2\lambda t}(1-e^{-\lambda t})^2 + 4e^{-\lambda t}(1-e^{-\lambda t})^3 + (1-e^{-\lambda t})^4 \Rightarrow$$

$$P(0\ \text{fail}) = e^{-4\lambda t},\ P(1\ \text{fail}) = 4e^{-3\lambda t}(1-e^{-\lambda t}),\ P(2\ \text{fail}) = 6e^{-2\lambda t}(1-e^{-\lambda t})^2,$$

$$P(3\ \text{fail}) = 4e^{-\lambda t}(1-e^{-\lambda t})^3,\ P(4\ \text{fail}) = (1-e^{-\lambda t})^4 \Rightarrow$$

$$P(4\ \text{of}\ 4) = P(0\ \text{fail}) = e^{-4\lambda t}$$

$$P(3\ \text{of}\ 4) = P(0\ \text{fail}) + P(1\ \text{fail}) = e^{-4\lambda t} + 4e^{-3\lambda t}(1-e^{-\lambda t})$$

$$P(2\ \text{of}\ 4) = P(0\ \text{fail}) + P(1\ \text{fail}) + P(2\ \text{fail}) = e^{-4\lambda t} + 4e^{-3\lambda t}(1-e^{-\lambda t}) + 6e^{-2\lambda t}(1-e^{-\lambda t})^2$$

$$P(1\ \text{of}\ 4) = e^{-4\lambda t} + 4e^{-3\lambda t}(1-e^{-\lambda t}) + 6e^{-2\lambda t}(1-e^{-\lambda t})^2 + 4e^{-\lambda t}(1-e^{-\lambda t})^3 \Rightarrow$$

Case n = 4 (MTTF)

$$MTTF\,(4\text{ of }4) = \int_0^\infty P(4\text{ of }4)\,dt = \int_0^\infty e^{-4\lambda t}\,dt = \frac{1}{4\lambda}$$

$$MTTF\,(3\text{ of }4) = \int_0^\infty P(3\text{ of }4)\,dt = \int_0^\infty e^{-4\lambda t}\,dt + 4\int_0^\infty e^{-3\lambda t}(1-e^{-\lambda t})\,dt = \frac{1}{4\lambda}+\frac{1}{3\lambda} = \frac{7}{12\lambda}$$

$$MTTR(2\text{ of }4) = \int_0^\infty P(2\text{ of }4)\,dt = \int_0^\infty e^{-4\lambda t}\,dt + 4\int_0^\infty e^{-3\lambda t}(1-e^{-\lambda t})\,dt + 6\int_0^\infty e^{-2\lambda t}(1-e^{-\lambda t})^2\,dt = \frac{1}{4\lambda}+\frac{1}{3\lambda}+\frac{1}{2\lambda} = \frac{13}{12\lambda}$$

$$MTTR(1\text{ of }4) = \int_0^\infty e^{-4\lambda t}\,dt + 4\int_0^\infty e^{-3\lambda t}(1-e^{-\lambda t})\,dt + 6\int_0^\infty e^{-2\lambda t}(1-e^{-\lambda t})^2\,dt + 4\int_0^\infty e^{-\lambda t}(1-e^{-\lambda t})^3\,dt = \frac{1}{4\lambda}+\frac{1}{3\lambda}+\frac{1}{2\lambda}+\frac{1}{\lambda} = \frac{25}{12\lambda}$$

Conclusion:

With the use of this table and/or using the simple and time-saving algorithm revealed here, the design engineer can quickly generate MTTF calculations. The engineer can quickly do "what if" analyses with respect to reliability. But the work must not end there. He or she must be very careful when adding redundancy to a design. Obviously, weight, size and cost will have to increase, and this must be factored into the decision-making process. Then, there is also the issue of built-in-test (BIT) that has to be addressed; possibly more important than cost are the weight, size and BIT issues. Will the increased weight and/or size have any degrading effects on safety? Will additional BIT have to be added? The effects on safety could be obvious, or they could be quite subtle. The design team must be very careful when assessing the overall effects of redundancy before any design change decisions are made.

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Theorems and Examples

Theorem 1

Theorem 1

If p = probability of success = $e^{-\lambda t}$ for one item, then

$$\text{MTTF}(m \text{ of } n)\text{ configuration} = \frac{1}{\lambda}\left[\frac{1}{n} + \frac{1}{(n-1)} + \frac{1}{(n-2)} + \cdots + \frac{1}{m}\right]$$

Proof:
Recall the binomial expansion

$$(p+q)^n = \binom{n}{0}p^n + \binom{n}{1}p^{n-1}q + \binom{n}{2}p^{n-2}q^2 + \cdots + \binom{n}{n}q^n \quad \text{where } \binom{n}{r} = nCr = \frac{n!}{(n-r)!\,r!}$$

The n+1 terms of the expansion reveal the probabilities of exactly 0, 1, 2 ··· n failures as follows:

$$p+q=1 \Rightarrow (p+q)^n = 1 = \binom{n}{0}p^n + \binom{n}{1}p^{n-1}q + \binom{n}{2}p^{n-2}q^2 + \cdots + \binom{n}{n}q^n \quad (1)$$

$$\Rightarrow P(\text{exactly } k \text{ failures}) = \binom{n}{k}p^{n-k}q^k = \binom{n}{k}e^{-(n-k)\lambda t}(1-e^{-\lambda t})^k \quad 0 \le k \le n$$

$$\text{Now} \int_0^\infty P(\text{exactly } k \text{ failures})\,dt = \int_0^\infty (k^{th} \text{ term above})\,dt = \binom{n}{k}\int_0^\infty e^{-(n-k)\lambda t}(1-e^{-\lambda t})^k\,dt$$

$$\text{Recall: } (1-a)^k = 1 - \binom{k}{1}a + \binom{k}{2}a^2 + \cdots + (-1)^{k-1}\binom{k}{k-1}a^{k-1} + (-1)^k a^k \Rightarrow$$

$$(1-e^{-\lambda t})^k = 1 - \binom{k}{1}e^{-\lambda t} + \binom{k}{2}e^{-2\lambda t} + \cdots + (-1)^{k-1}\binom{k}{k-1}e^{-(k-1)\lambda t} + (-1)^k e^{-k\lambda t} \Rightarrow$$

$$\int_0^\infty P(\text{exactly } k \text{ failures})\,dt = \binom{n}{k}\int_0^\infty e^{-(n-k)\lambda t}\left[1 - \binom{k}{1}e^{-\lambda t} - \binom{k}{2}e^{-2\lambda t} + \cdots + (-1)^{k-1}\binom{k}{k-1}e^{-(k-1)\lambda t} + (-1)^k e^{-k\lambda t}\right]dt$$

$$= \binom{n}{k}\int_0^\infty \left[e^{-(n-k)\lambda t} - \binom{k}{1}e^{-(n-k+1)\lambda t} - \binom{k}{2}e^{-(n-k+2)\lambda t} + \cdots + (-1)^{k-1}\binom{k}{k-1}e^{-(n-1)\lambda t} - (-1)^k e^{-n\lambda t}\right]dt$$

$$= \frac{1}{\lambda}\binom{n}{k}\left[\frac{1}{(n-k)} - \binom{k}{1}\frac{1}{(n-k+1)} + \binom{k}{2}\frac{1}{(n-k+2)} - \cdots + (-1)^{k-1}\binom{k}{k-1}\frac{1}{(n-1)} + (-1)^k\frac{1}{n}\right]$$

$$= \frac{1}{\lambda}\binom{n}{k}\cdot\frac{1}{(n-k)\cdot\binom{n}{k}} = \frac{1}{\lambda}\cdot\frac{1}{(n-k)} \text{ for each term by corollary to Theorem 2} \Rightarrow$$

$$\text{Now } P(m \text{ of } n) = \sum_{k=0}^{n-m}\binom{n}{k}e^{-(n-k)\lambda t}(1-e^{-\lambda t})^k \quad 0 \le m \le n \Rightarrow$$

$$\text{MTTF}(m \text{ of } n)\text{ configuration} = \int_0^\infty P(m \text{ of } n)\,dt = \int_0^\infty\left(\sum_{k=0}^{n-m}\binom{n}{k}e^{-(n-k)\lambda t}(1-e^{-\lambda t})^k\right)dt$$

$$= \sum_{k=0}^{n-m}\left(\binom{n}{k}\int_0^\infty e^{-(n-k)\lambda t}(1-e^{-\lambda t})^k\,dt\right) = \sum_{k=0}^{n-m}\frac{1}{\lambda}\cdot\frac{1}{(n-k)} = \frac{1}{\lambda}\sum_{k=0}^{n-m}\frac{1}{(n-k)}$$

$$= \frac{1}{\lambda}\left(\frac{1}{n} + \frac{1}{(n-1)} + \frac{1}{(n-2)} + \cdots + \frac{1}{m}\right) \quad \text{//}$$

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Theorem 2

Theorem 2

$$\frac{1}{n}\binom{m}{0}-\frac{1}{n-1}\binom{m}{1}+\frac{(-1)^2}{n-2}\binom{m}{2}+\cdots+\frac{(-1)^{m-1}}{n-m+1}\binom{m}{m-1}+\frac{(-1)^m}{n-m}\binom{m}{m} \quad (1)$$

$$=\frac{(-1)^m m!}{n\cdot(n-1)\cdot(n-2)\cdot\cdots\cdot(n-m)} \text{ or } \frac{(-1)^m}{(n-m)\cdot\binom{n}{m}} \quad \text{where } \binom{n}{r}=\frac{n!}{(n-r)!\cdot r!}$$

Proof by building block induction:

Starting with $m=2 \Rightarrow \frac{1}{n}-\frac{2}{n-1}+\frac{1}{n-2}=\frac{2}{n\cdot(n-1)\cdot(n-2)}$ by identity for $n\ge 3$

Now $\frac{1}{n}-\frac{3}{n-1}+\frac{3}{n-2}-\frac{1}{n-3}=\left(\frac{1}{n}-\frac{2}{n-1}+\frac{1}{n-2}\right)-\left(\frac{1}{n-1}-\frac{2}{n-2}+\frac{1}{n-3}\right)$ by identity

$$\Rightarrow \frac{1}{n}-\frac{3}{n-1}+\frac{3}{n-2}-\frac{1}{n-3}=\frac{2}{n\cdot(n-1)\cdot(n-2)}-\frac{2!}{(n-1)\cdot(n-2)\cdot(n-3)}$$

$$=\frac{2!\cdot(n-3)}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)}-\frac{2!\cdot n}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)}=\frac{-3!}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)} \text{ for } n\ge 4$$

so Theorem is proven for $m=3$

Similarly $\frac{1}{n}-\frac{4}{n-1}+\frac{6}{n-2}-\frac{4}{n-3}+\frac{1}{n-4}$

$$=\left(\frac{1}{n}-\frac{3}{n-1}+\frac{3}{n-2}-\frac{1}{n-3}\right)-\left(\frac{1}{n-1}-\frac{3}{n-2}+\frac{3}{n-3}-\frac{1}{n-4}\right) \text{ by identity}$$

$$=\frac{-3!}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)}-\frac{-3!}{(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)} \text{ (from above)}$$

$$=\frac{-3!\cdot(n-4)+3!\cdot n}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)}=\frac{4!}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)}$$

so Theorem is proven for $m=4$

Similarly $\frac{1}{n}-\frac{5}{n-1}+\frac{10}{n-2}-\frac{10}{n-3}+\frac{5}{n-4}-\frac{1}{n-5}$

$$=\left(\frac{1}{n}-\frac{4}{n-1}+\frac{6}{n-2}-\frac{4}{n-3}+\frac{1}{n-4}\right)-\left(\frac{1}{n-1}-\frac{4}{n-2}+\frac{6}{n-3}-\frac{4}{n-4}+\frac{1}{n-5}\right)$$

$$=\frac{4!}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)}-\frac{4!}{(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)\cdot(n-5)}$$

$$=\frac{4!\cdot(n-5)-4!\cdot n}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)\cdot(n-5)}=\frac{-5!}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)\cdot(n-5)}$$

so Theorem is proven for $m=5$
and similarly for $m\ge 6$ //

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Corollary to Theorem 2

Corollary to Theorem 2

$$\frac{1}{n-m}\binom{m}{0}-\frac{1}{n-m+1}\binom{m}{1}+\frac{(-1)^2}{n-m+2}\binom{m}{2}+\cdots+\frac{(-1)^{m-1}}{n-1}\binom{m}{m-1}+\frac{(-1)^m}{n}\binom{m}{m} \quad (2)$$

$$=\frac{m!}{n\cdot(n-1)\cdot(n-2)\cdot\cdots\cdot(n-m)} \text{ or } \frac{1}{(n-m)\cdot\binom{n}{m}}$$

Proof:
Case for m even, note Equation (1) = Equation (2) by observation

$$(2)=\frac{1}{n-m}\binom{m}{0}-\frac{1}{n-m+1}\binom{m}{1}+\cdots+\frac{(-1)^{m-2}}{n-2}\binom{m}{m-2}+\frac{(-1)^{m-1}}{n-1}\binom{m}{m-1}+\frac{(-1)^m}{n}\binom{m}{m}$$

$$(1)=\frac{(-1)^m}{n-m}\binom{m}{m}+\frac{(-1)^{m-1}}{n-m+1}\binom{m}{m-1}+\cdots+\frac{(-1)^2}{n-2}\binom{m}{2}-\frac{1}{n-1}\binom{m}{1}+\frac{1}{n}\binom{m}{0}$$

$$=\frac{(-1)^m}{(n-m)\cdot\binom{n}{m}}=\frac{1}{(n-m)\cdot\binom{n}{m}}$$

Case for m odd, each term of Equation (2) has the reverse sign of Equation (1) by observation

$$(2)=\frac{1}{n-m}\binom{m}{0}-\frac{1}{n-m+1}\binom{m}{1}+\cdots+\frac{(-1)^{m-2}}{n-2}\binom{m}{m-2}+\frac{(-1)^{m-1}}{n-1}\binom{m}{m-1}+\frac{(-1)^m}{n}\binom{m}{m}$$

$$(1)=\frac{(-1)^m}{n-m}\binom{m}{m}+\frac{(-1)^{m-1}}{n-m+1}\binom{m}{m-1}+\cdots+\frac{(-1)^2}{n-2}\binom{m}{2}-\frac{1}{n-1}\binom{m}{1}+\frac{1}{n}\binom{m}{0}$$

$$\Rightarrow \text{Equation}(2)=(-1)\cdot\text{Equation}(1)=\frac{(-1)\cdot(-1)^m}{(n-m)\cdot\binom{n}{m}}=\frac{1}{(n-m)\cdot\binom{n}{m}} \quad //$$

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Examples of Theorem 2

Examples of Theorem 2

$$n=5,\ m=2 \Rightarrow 1\cdot\frac{1}{5}-2\cdot\frac{1}{4}+1\cdot\frac{1}{3}=\frac{(-1)^2}{3\cdot\binom{5}{2}}=\frac{(-1)^2 2!}{3\cdot 4\cdot 5}=\frac{2!}{3\cdot 4\cdot 5}$$

$$n=5,\ m=3 \Rightarrow 1\cdot\frac{1}{5}-3\cdot\frac{1}{4}+3\cdot\frac{1}{3}-1\cdot\frac{1}{2}=\frac{(-1)^3}{2\cdot\binom{5}{3}}=\frac{(-1)^3 3!}{2\cdot 3\cdot 4\cdot 5}=\frac{-3!}{2\cdot 3\cdot 4\cdot 5}$$

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Examples of Corollary to Theorem 2

Examples of Corollary to Theorem 2

$$n=5,\ m=2 \Rightarrow 1\cdot\frac{1}{3}-2\cdot\frac{1}{4}+1\cdot\frac{1}{5}=\frac{1}{3\cdot\binom{5}{2}}=\frac{2!}{3\cdot 4\cdot 5}$$

$$n=5,\ m=3 \Rightarrow 1\cdot\frac{1}{2}-3\cdot\frac{1}{3}+3\cdot\frac{1}{4}-1\cdot\frac{1}{5}=\frac{1}{2\cdot\binom{5}{3}}=\frac{3!}{2\cdot 3\cdot 4\cdot 5}$$

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In the Spotlight

Redundancy for Safety

by Vito Faraci Jr.

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Lemma 1

Lemma 1

If $A=\left[\frac{1}{n}\binom{m}{0}-\frac{1}{n-1}\binom{m}{1}+\frac{(-1)^2}{n-2}\binom{m}{2}+\cdots+\frac{(-1)^{m-1}}{n-m+1}\binom{m}{m-1}+\frac{(-1)^m}{n-m}\binom{m}{m}\right]$ and

$B=\left[\frac{1}{n-1}\binom{m}{0}-\frac{1}{n-2}\binom{m}{1}+\frac{(-1)^2}{n-3}\binom{m}{2}+\cdots+\frac{(-1)^{m-1}}{n-m}\binom{m}{m-1}+\frac{(-1)^m}{n-m-1}\binom{m}{m}\right]$ then

$$A-B=\frac{1}{n}\binom{m+1}{0}-\frac{1}{n-1}\binom{m+1}{1}+\frac{(-1)^2}{n-2}\binom{m+1}{2}+\cdots+\frac{(-1)^m}{n-m}\binom{m+1}{m}+\frac{(-1)^{m+1}}{n-m-1}\binom{m+1}{m+1}$$

Proof:
Rearranging and aligning like terms $\Rightarrow$

$$A=\left[-\frac{1}{n-1}\binom{m}{1}+\frac{(-1)^2}{n-2}\binom{m}{2}+\cdots+\frac{(-1)^{m-1}}{n-m+1}\binom{m}{m-1}+\frac{(-1)^m}{n-m}\binom{m}{m}+\frac{1}{n}\binom{m}{0}\right]$$

$$B=\left[\frac{1}{n-1}\binom{m}{0}-\frac{1}{n-2}\binom{m}{1}+\frac{(-1)^2}{n-3}\binom{m}{2}+\cdots+\frac{(-1)^{m-1}}{n-m}\binom{m}{m-1}+\frac{(-1)^m}{n-m-1}\binom{m}{m}\right]$$

Subtracting like terms for $k=1$ to $m \Rightarrow$

$$\frac{(-1)^k}{n-k}\binom{m}{k}(\text{from }A)-\frac{(-1)^{k-1}}{n-k}\binom{m}{k-1}(\text{from }B)=\frac{(-1)^k}{n-k}\binom{m}{k}+\frac{(-1)^k}{n-k}\binom{m}{k-1}$$

$$=\frac{(-1)^k}{n-k}\left[\binom{m}{k}+\binom{m}{k-1}\right]=\frac{(-1)^k}{n-k}\binom{m+1}{k} \text{ by Lemma 2 } //$$

Example of Lemma 1

$$\left(\frac{1}{n}-\frac{3}{n-1}+\frac{3}{n-2}-\frac{1}{n-3}\right)-\left(\frac{1}{n-1}-\frac{3}{n-2}+\frac{3}{n-3}-\frac{1}{n-4}\right)$$

$$=\frac{1}{n}-\frac{4}{n-1}+\frac{6}{n-2}-\frac{4}{n-3}+\frac{1}{n-4}$$

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Lemma 2

Lemma 2

$\binom{n+1}{r}=\binom{n}{r}+\binom{n}{r-1}$ for $1 \le r \le n$ i.e. for all terms not equal to 1

Proof :

$$\frac{(n+1)!}{(n+1-r)!\,r!}=\frac{n!}{(n-r)!\,r!}+\frac{n!}{[n-(1-r)]!\,(r-1)!}\Leftrightarrow$$

$$\frac{(n+1)!}{(n+1-r)!\,r!}=\frac{n!}{(n-r)!\,r!}+\frac{n!}{(n+1-r)!\,(r-1)!}\Leftrightarrow$$

$$\frac{n+1}{(n+1-r)!\,r!}=\frac{1}{(n-r)!\,r!}+\frac{1}{(n+1-r)!\,(r-1)!}\Leftrightarrow$$

$$\frac{n+1}{(n+1-r)!}=\frac{1}{(n-r)!}+\frac{r}{(n+1-r)!}\Leftrightarrow n+1=\frac{(n+1-r)!}{(n-r)!}+r\Leftrightarrow$$

$$n+1-r=\frac{(n+1-r)!}{(n-r)!}\Leftrightarrow n+1-r=n+1-r \; //$$

Examples of Lemma 2

$$\binom{5}{1}=\binom{4}{1}+\binom{4}{0} \quad \binom{5}{2}=\binom{4}{2}+\binom{4}{1} \quad \binom{5}{3}=\binom{4}{3}+\binom{4}{2} \quad \binom{5}{4}=\binom{4}{4}+\binom{4}{3}$$

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About the Author

Vito Faraci is a mathematician by education, and an electrical engineer by trade. He has 20 years of experience with qualitative and quantitative analyses of reliability, built-in-test and safety-related events. He has also served as a reliability and Markov analysis consultant for the Federal Aviation Administration and commercial airlines. He has given lectures and seminars in the United States and Canada on the subject of calculating probability of failure of electrical and mechanical components and systems. Faraci has also served as an adjunct math professor at New York Institute of Technology.

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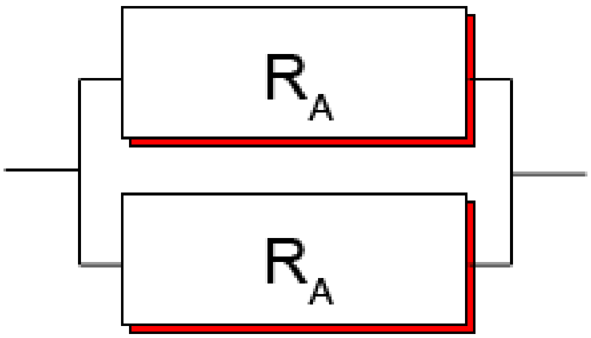
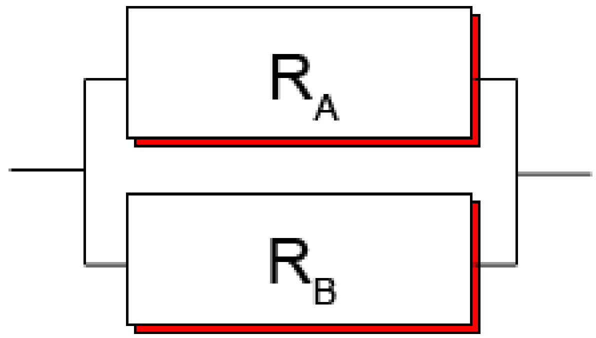
Component λs Equal	Component λs Not Equal
	
$R = 1 - \left[(1 - R_A)^2\right]$	$R = 1 - \left[(1 - R_A)(1 - R_B)\right]$

R = Reliability of Configuration, R_A = Reliability of Item A, R_B = Reliability of Item B

$\lambda_{system} = \dfrac{2\lambda_A{}^3}{3\lambda_A{}^2} = \dfrac{2\lambda_A}{3}$	$\lambda_{system} = \dfrac{\lambda_A \lambda_B (\lambda_A + \lambda_B)}{(\lambda_A + \lambda_B)^2 - \lambda_A \lambda_B}$

λ_A = Failure Rate of Item A, λ_B = Failure Rate of Item B

m of n n = 2	System Probability of Success (P_s)	System (MTTF) (by Theorem 1 in Theorems and Examples)	FR
2 of 2	$e^{-2\lambda t}$	$\frac{1}{\lambda}\left(\frac{1}{2}\right)$	2λ
1 of 2	$e^{-2\lambda t} + 2(e^{-\lambda t} - e^{-2\lambda t})$	$\frac{1}{\lambda}\left(\frac{1}{2}+1\right) = \frac{3}{2\lambda}$	$\frac{2\lambda}{3}$
n = 3			
3 of 3	$e^{-3\lambda t}$	$\frac{1}{\lambda}\left(\frac{1}{3}\right)$	3λ
2 of 3	$e^{-3\lambda t} + 3e^{-2\lambda t}(1-e^{-\lambda t})$	$\frac{1}{\lambda}\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{5}{6\lambda}$	$\frac{6\lambda}{5}$
1 of 3	$e^{-3\lambda t} + 3e^{-2\lambda t}(1-e^{-\lambda t}) + 3e^{-\lambda t}(1-e^{-\lambda t})^2$	$\frac{1}{\lambda}\left(\frac{1}{3}+\frac{1}{2}+1\right) = \frac{11}{6\lambda}$	$\frac{6\lambda}{11}$
n = 4			
4 of 4	$e^{-4\lambda t}$	$\frac{1}{\lambda}\left(\frac{1}{4}\right)$	4λ
3 of 4	$e^{-4\lambda t} + 4e^{-3\lambda t}(1-e^{-\lambda t})$	$\frac{1}{\lambda}\left(\frac{1}{4}+\frac{1}{3}\right) = \frac{7}{12\lambda}$	$\frac{12\lambda}{7}$
2 of 4	$e^{-4\lambda t} + 4e^{-3\lambda t}(1-e^{-\lambda t}) + 6e^{-2\lambda t}(1-e^{-\lambda t})^2$	$\frac{1}{\lambda}\left(\frac{1}{4}+\frac{1}{3}+\frac{1}{2}\right) = \frac{13}{12\lambda}$	$\frac{12\lambda}{13}$
1 of 4	$e^{-4\lambda t} + 4e^{-3\lambda t}(1-e^{-\lambda t}) + 6e^{-2\lambda t}(1-e^{-\lambda t})^2$ $+ 4e^{-\lambda t}(1-e^{-\lambda t})^3$	$\frac{1}{\lambda}\left(\frac{1}{4}+\frac{1}{3}+\frac{1}{2}+1\right) = \frac{25}{12\lambda}$	$\frac{12\lambda}{25}$
n = 5			
5 of 5	$e^{-5\lambda t}$	$\frac{1}{\lambda}\left(\frac{1}{5}\right)$	5λ
4 of 5	$e^{-5\lambda t} + 5e^{-4\lambda t}(1-e^{-\lambda t})$	$\frac{1}{\lambda}\left(\frac{1}{5}+\frac{1}{4}\right) = \frac{9}{20\lambda}$	$\frac{20\lambda}{9}$
3 of 5	$e^{-5\lambda t} + 5e^{-4\lambda t}(1-e^{-\lambda t}) + 10e^{-3\lambda t}(1-e^{-\lambda t})^2$	$\frac{1}{\lambda}\left(\frac{1}{5}+\frac{1}{4}+\frac{1}{3}\right) = \frac{47}{60\lambda}$	$\frac{60\lambda}{47}$
2 of 5	$e^{-5\lambda t} + 5e^{-4\lambda t}(1-e^{-\lambda t}) + 10e^{-3\lambda t}(1-e^{-\lambda t})^2$ $+ 10e^{-2\lambda t}(1-e^{-\lambda t})^3$	$\frac{1}{\lambda}\left(\frac{1}{5}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2}\right) = \frac{77}{60\lambda}$	$\frac{60\lambda}{77}$
1 of 5	$e^{-5\lambda t} + 5e^{-4\lambda t}(1-e^{-\lambda t}) + 10e^{-3\lambda t}(1-e^{-\lambda t})^2$ $+ 10e^{-2\lambda t}(1-e^{-\lambda t})^3 + 5e^{-\lambda t}(1-e^{-\lambda t})^4$	$\frac{1}{\lambda}\left(\frac{1}{5}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2}+1\right) = \frac{137}{60\lambda}$	$\frac{60\lambda}{137}$
m of n	$1 \le m \le n.$		
P_s	$\sum_{k=0}^{m} \binom{n}{k} e^{-(n-k)\lambda t}(1-e^{-\lambda t})^k$		
MTTF	$\frac{1}{\lambda}\left(\frac{1}{n}+\frac{1}{(n-1)}+\frac{1}{(n-2)}+\cdots+\frac{1}{(m+1)}+\frac{1}{m}\right)$		

[illegible]	$\frac{[illegible]}{\lambda}\left(\frac{[illegible]}{n}+\frac{[illegible]}{(n-1)}+\frac{[illegible]}{(n-2)}+\dots+\frac{[illegible]}{(m+1)}+\frac{[illegible]}{m}\right)$	

Theorem 1

If $p = \text{probability of success} = e^{-\lambda t}$ for one item, then

$$\text{MTTF (m of n) configuration} = \frac{1}{\lambda}\left(\frac{1}{n} + \frac{1}{(n-1)} + \frac{1}{(n-2)} + \cdots + \frac{1}{m}\right)$$

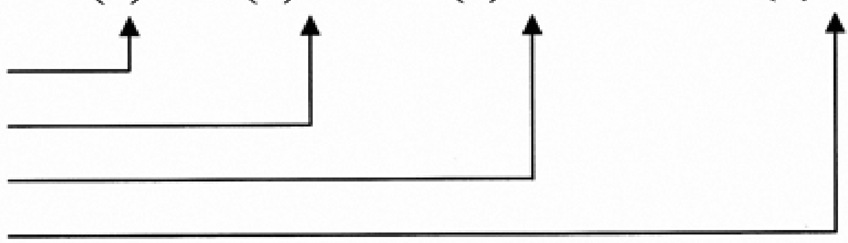
Proof:

Recall the binomial expansion

$$(p+q)^n = \binom{n}{0}p^n + \binom{n}{1}p^{n-1}q + \binom{n}{2}p^{n-2}q^2 + \cdots + \binom{n}{n}q^n \quad \text{where } \binom{n}{r} = nCr = \frac{n!}{(n-r)!\cdot r!}$$

The n+1 terms of the expansion reveal the probabilities of exactly 0, 1, 2 ··· n failures as follows:

$$p+q = 1 \Rightarrow (p+q)^n = 1 = \binom{n}{0}p^n + \binom{n}{1}p^{n-1}q + \binom{n}{2}p^{n-2}q^2 + \cdots + \binom{n}{n}q^n \qquad (1)$$



$$\Rightarrow P(\text{exactly k failures}) = \binom{n}{k}p^{n-k}q^k = \binom{n}{k}e^{-(n-k)\lambda t}(1-e^{-\lambda t})^k \quad 0 \le k \le n$$

$$\text{Now} \quad \int_0^\infty P(\text{exactly k failures})\,dt = \int_0^\infty (k^{th} \text{ term above})\,dt = \binom{n}{k}\int_0^\infty e^{-(n-k)\lambda t}(1-e^{-\lambda t})^k\,dt$$

$$\text{Recall } (1-a)^k = 1 - \binom{k}{1}a + \binom{k}{2}a^2 + \cdots + (-1)^{k-1}\binom{k}{k-1}a^{k-1} + (-1)^k a^k \Rightarrow$$

$$(1-e^{-\lambda t})^k = 1 - \binom{k}{1}e^{-\lambda t} + \binom{k}{2}e^{-2\lambda t} + \cdots + (-1)^{k-1}\binom{k}{k-1}e^{-(k-1)\lambda t} + (-1)^k e^{-k\lambda t} \Rightarrow$$

$$\int_0^\infty P(\text{exactly k failures})\,dt = \binom{n}{k}\int_0^\infty e^{-(n-k)\lambda t}\left[1 - \binom{k}{1}e^{-\lambda t} + \binom{k}{2}e^{-2\lambda t} + \cdots + (-1)^{k-1}\binom{k}{k-1}e^{-(k-1)\lambda t} + (-1)^k e^{-k\lambda t}\right]dt$$

$$= \binom{n}{k}\int_0^\infty \left[e^{-(n-k)\lambda t} - \binom{k}{1}e^{-(n-k+1)\lambda t} + \binom{k}{2}e^{-(n-k+2)\lambda t} + \cdots + (-1)^{k-1}\binom{k}{k-1}e^{-(n-1)\lambda t} + (-1)^k e^{-n\lambda t}\right]dt$$

$$= \frac{1}{\lambda}\cdot\binom{n}{k}\cdot\left[\frac{1}{(n-k)} - \binom{k}{1}\frac{1}{(n-k+1)} + \binom{k}{2}\frac{1}{(n-k+2)} + \cdots + (-1)^{k-1}\binom{k}{k-1}\frac{1}{(n-1)} + (-1)^k\frac{1}{n}\right]$$

$$= \frac{1}{\lambda}\cdot\binom{n}{k}\cdot\frac{1}{(n-k)\cdot\binom{n}{k}} = \frac{1}{\lambda}\cdot\frac{1}{(n-k)} \quad \text{for each term by corollary to Theorem 2} \Rightarrow$$

$$\lambda \binom{k}{} \quad (n-k)\cdot\binom{n}{k} \quad \lambda \; (n-k)$$

$$\text{Now } P(m \text{ of } n) = \sum_{k=0}^{m}\binom{n}{k} e^{-(n-k)\lambda t}(1-e^{-\lambda t})^{k} \qquad 0 \le m \le n \;\Rightarrow$$

$$\text{MTTF}(m \text{ of } n)\text{configuration} = \int_0^{\infty} P(m \text{ of } n)\,dt = \int_0^{\infty}\left(\sum_{k=0}^{m}\binom{n}{k} e^{-(n-k)\lambda t}(1-e^{-\lambda t})^{k}\right)dt$$

$$= \sum_{k=0}^{m}\left(\binom{n}{k}\int_0^{\infty} e^{-(n-k)\lambda t}(1-e^{-\lambda t})^{k}\,dt\right) = \sum_{k=0}^{m}\frac{1}{\lambda}\cdot\frac{1}{(n-k)} = \frac{1}{\lambda}\sum_{k=0}^{m}\frac{1}{(n-k)}$$

$$= \frac{1}{\lambda}\left(\frac{1}{n} + \frac{1}{(n-1)} + \frac{1}{(n-2)} + \cdots + \frac{1}{m}\right) \;//$$

Theorem 2

$$\frac{1}{n}\cdot\binom{m}{0} - \frac{1}{n-1}\cdot\binom{m}{1} + \frac{(-1)^2}{n-2}\binom{m}{2} + \cdots + \frac{(-1)^{m-1}}{n-m+1}\binom{m}{m-1} + \frac{(-1)^m}{n-m}\binom{m}{m} \quad (1)$$

$$= \frac{(-1)^m m!}{n\cdot(n-1)\cdot(n-2)\cdot \cdots \cdot (n-m)} \quad \text{or} \quad \frac{(-1)^m}{(n-m)\cdot\binom{n}{m}} \quad \text{where } \binom{n}{r} = \frac{n!}{(n-r)!\cdot r!}$$

Proof by building block induction:

Starting with $m = 2 \Rightarrow \frac{1}{n} - \frac{2}{n-1} + \frac{1}{n-2} = \frac{2!}{n\cdot(n-1)\cdot(n-2)}$ by identity for $n \geq 3$

Now $\frac{1}{n} - \frac{3}{n-1} + \frac{3}{n-2} - \frac{1}{n-3} = \left(\frac{1}{n} - \frac{2}{n-1} + \frac{1}{n-2}\right) - \left(\frac{1}{n-1} - \frac{2}{n-2} + \frac{1}{n-3}\right)$ by identity

$$\Rightarrow \frac{1}{n} - \frac{3}{n-1} + \frac{3}{n-2} - \frac{1}{n-3} = \frac{2!}{n\cdot(n-1)\cdot(n-2)} - \frac{2!}{(n-1)\cdot(n-2)\cdot(n-3)}$$

$$= \frac{2!\cdot(n-3)}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)} - \frac{2!\cdot n}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)} = \frac{-3!}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)} \quad \text{for } n \geq 4$$

so Theorem is proven for $m = 3$

Similarly $\frac{1}{n} - \frac{4}{n-1} + \frac{6}{n-2} - \frac{4}{n-3} + \frac{1}{n-4}$

$= \left(\frac{1}{n} - \frac{3}{n-1} + \frac{3}{n-2} - \frac{1}{n-3}\right) - \left(\frac{1}{n-1} - \frac{3}{n-2} + \frac{3}{n-3} - \frac{1}{n-4}\right)$ by identity

$= \frac{-3!}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)} - \frac{-3!}{(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)}$ (from above)

$$= \frac{-3!\cdot(n-4) + 3!\cdot n}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)} = \frac{4!}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)}$$

so Theorem is proven for $m = 4$

Similarly $\frac{1}{n} - \frac{5}{n-1} + \frac{10}{n-2} - \frac{10}{n-3} + \frac{5}{n-4} - \frac{1}{n-5}$

$$= \left(\frac{1}{n} - \frac{4}{n-1} + \frac{6}{n-2} - \frac{4}{n-3} + \frac{1}{n-4}\right) - \left(\frac{1}{n-1} - \frac{4}{n-2} + \frac{6}{n-3} - \frac{4}{n-4} + \frac{1}{n-5}\right)$$

$$= \frac{4!}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)} - \frac{4!}{(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)\cdot(n-5)}$$

$$= \frac{4!\cdot(n-5) - 4!\cdot n}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)\cdot(n-5)} = \frac{-5!}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)\cdot(n-5)}$$

so Theorem is proven for $m = 5$

and similarly for $m \geq 6$ //

so Theorem is proven for $m = 5$
and similarly for $m \geq 6$ //

Corollary to Theorem 2

$$\frac{1}{n-m}\cdot\binom{m}{0}-\frac{1}{n-m+1}\cdot\binom{m}{1}+\frac{(-1)^2}{n-m+2}\cdot\binom{m}{2}+\cdots+\frac{(-1)^{m-1}}{n-1}\cdot\binom{m}{m-1}+\frac{(-1)^m}{n}\cdot\binom{m}{m} \quad (2)$$

$$=\frac{m!}{n\cdot(n-1)\cdot(n-2)\cdot\,\cdots\,\cdot(n-m)} \text{ or } \frac{1}{(n-m)\cdot\binom{n}{m}}$$

Proof:

Case for m even, note Equation (1) = Equation (2) by observation

$$(2)=\frac{1}{n-m}\cdot\binom{m}{0}-\frac{1}{n-m+1}\cdot\binom{m}{1}+\cdots+\frac{(-1)^{m-2}}{n-2}\cdot\binom{m}{m-2}+\frac{(-1)^{m-1}}{n-1}\cdot\binom{m}{m-1}+\frac{(-1)^m}{n}\cdot\binom{m}{m}$$

$$(1)=\frac{(-1)^m}{n-m}\cdot\binom{m}{m}+\frac{(-1)^{m-1}}{n-m+1}\cdot\binom{m}{m-1}+\cdots+\frac{(-1)^2}{n-2}\cdot\binom{m}{2}-\frac{1}{n-1}\cdot\binom{m}{1}+\frac{1}{n}\cdot\binom{m}{0}$$

$$=\frac{(-1)^m}{(n-m)\cdot\binom{n}{m}}=\frac{1}{(n-m)\cdot\binom{n}{m}}$$

Case for m odd, each term of Equation (2) has the reverse sign of Equation (1) by observation

$$(2)=\frac{1}{n-m}\cdot\binom{m}{0}-\frac{1}{n-m+1}\cdot\binom{m}{1}+\cdots+\frac{(-1)^{m-2}}{n-2}\cdot\binom{m}{m-2}+\frac{(-1)^{m-1}}{n-1}\cdot\binom{m}{m-1}+\frac{(-1)^m}{n}\cdot\binom{m}{m}$$

$$(1)=\frac{(-1)^m}{n-m}\cdot\binom{m}{m}+\frac{(-1)^{m-1}}{n-m+1}\cdot\binom{m}{m-1}+\cdots+\frac{(-1)^2}{n-2}\cdot\binom{m}{2}-\frac{1}{n-1}\cdot\binom{m}{1}+\frac{1}{n}\cdot\binom{m}{0}$$

$$\Rightarrow \text{Equation}(2)=(-1)\cdot\text{Equation}(1)=\frac{(-1)\cdot(-1)^m}{(n-m)\cdot\binom{n}{m}}=\frac{1}{(n-m)\cdot\binom{n}{m}} \quad //$$

Examples of Theorem 2

$n = 5,\ m = 2 \ \Rightarrow\ 1\cdot\frac{1}{5} - 2\cdot\frac{1}{4} + 1\cdot\frac{1}{3} = \frac{(-1)^2}{3\cdot\binom{5}{2}} = \frac{(-1)^2\,2!}{3\cdot4\cdot5} = \frac{2!}{3\cdot4\cdot5}$
$n = 5,\ m = 3 \ \Rightarrow\ 1\cdot\frac{1}{5} - 3\cdot\frac{1}{4} + 3\cdot\frac{1}{3} - 1\cdot\frac{1}{2} = \frac{(-1)^3}{2\cdot\binom{5}{3}} = \frac{(-1)^3\,3!}{2\cdot3\cdot4\cdot5} = \frac{-3!}{2\cdot3\cdot4\cdot5}$

Examples of Corollary to Theorem 2

$n=5,\ m=2 \ \Rightarrow\ 1\cdot\frac{1}{3} - 2\cdot\frac{1}{4} + 1\cdot\frac{1}{5} = \frac{1}{3\cdot\binom{5}{2}} = \frac{2!}{3\cdot4\cdot5}$
$n=5,\ m=3 \ \Rightarrow\ 1\cdot\frac{1}{2} - 3\cdot\frac{1}{3} + 3\cdot\frac{1}{4} - 1\cdot\frac{1}{5} = \frac{1}{2\cdot\binom{5}{3}} = \frac{3!}{2\cdot3\cdot4\cdot5}$

Lemma 1

If $A=\left[\frac{1}{n}\cdot\binom{m}{0}-\frac{1}{n-1}\cdot\binom{m}{1}+\frac{(-1)^2}{n-2}\cdot\binom{m}{2}+\cdots+\frac{(-1)^{m-1}}{n-m+1}\cdot\binom{m}{m-1}+\frac{(-1)^m}{n-m}\cdot\binom{m}{m}\right]$ and

$B=\left[\frac{1}{n-1}\cdot\binom{m}{0}-\frac{1}{n-2}\cdot\binom{m}{1}+\frac{(-1)^2}{n-3}\cdot\binom{m}{2}+\cdots+\frac{(-1)^{m-1}}{n-m}\cdot\binom{m}{m-1}+\frac{(-1)^m}{n-m-1}\cdot\binom{m}{m}\right]$ then

$$A-B=\frac{1}{n}\cdot\binom{m+1}{0}-\frac{1}{n-1}\cdot\binom{m+1}{1}+\frac{(-1)^2}{n-2}\cdot\binom{m+1}{2}+\cdots+\frac{(-1)^m}{n-m}\cdot\binom{m+1}{m}+\frac{(-1)^{m+1}}{n-m-1}\cdot\binom{m+1}{m+1}$$

Proof:

Rearranging and aligning like terms $\Rightarrow$

$$A=\left[-\frac{1}{n-1}\cdot\binom{m}{1}+\frac{(-1)^2}{n-2}\cdot\binom{m}{2}+\cdots+\frac{(-1)^{m-1}}{n-m+1}\cdot\binom{m}{m-1}+\frac{(-1)^m}{n-m}\cdot\binom{m}{m}+\frac{1}{n}\cdot\binom{m}{0}\right]$$

$$B=\left[\frac{1}{n-1}\cdot\binom{m}{0}-\frac{1}{n-2}\cdot\binom{m}{1}+\frac{(-1)^2}{n-3}\cdot\binom{m}{2}+\cdots+\frac{(-1)^{m-1}}{n-m}\cdot\binom{m}{m-1}+\frac{(-1)^m}{n-m-1}\cdot\binom{m}{m}\right]$$

Subtracting like terms for $k=1$ to $m\Rightarrow$

$$\frac{(-1)^k}{n-k}\cdot\binom{m}{k}(\text{from A})-\frac{(-1)^{k-1}}{n-k}\cdot\binom{m}{k-1}(\text{from B})=\frac{(-1)^k}{n-k}\cdot\binom{m}{k}+\frac{(-1)^k}{n-k}\cdot\binom{m}{k-1}$$

$$=\frac{(-1)^k}{n-k}\cdot\left[\binom{m}{k}+\binom{m}{k-1}\right]=\frac{(-1)^k}{n-k}\cdot\binom{m+1}{k}\text{ by Lemma 2}\quad //$$

Example of Lemma 1

$$\left(\frac{1}{n}-\frac{3}{n-1}+\frac{3}{n-2}-\frac{1}{n-3}\right)-\left(\frac{1}{n-1}-\frac{3}{n-2}+\frac{3}{n-3}-\frac{1}{n-4}\right)$$

$$=\frac{1}{n}-\frac{4}{n-1}+\frac{6}{n-2}-\frac{4}{n-3}+\frac{1}{n-4}$$

Lemma 2

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1} \quad \text{for } 1 \le r \le n \quad \text{i.e. for all terms not equal to 1}$$

Proof :

$$\frac{(n+1)!}{(n+1-r)!\,r!} = \frac{n!}{(n-r)!\,r!} + \frac{n!}{[n-(1-r)]!\;(r-1)!} \Leftrightarrow$$

$$\frac{(n+1)!}{(n+1-r)!\,r!} = \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n+1-r)!\;(r-1)!} \Leftrightarrow$$

$$\frac{n+1}{(n+1-r)!\,r!} = \frac{1}{(n-r)!\,r!} + \frac{1}{(n+1-r)!\;(r-1)!} \Leftrightarrow$$

$$\frac{n+1}{(n+1-r)!} = \frac{1}{(n-r)!} + \frac{r}{(n+1-r)!} \Leftrightarrow n+1 = \frac{(n+1-r)!}{(n-r)!} + r \Leftrightarrow$$

$$n+1-r = \frac{(n+1-r)!}{(n-r)!} \Leftrightarrow n+1-r = n+1-r \; //$$

Examples of Lemma 2

$$\binom{5}{1} = \binom{4}{1} + \binom{4}{0} \qquad \binom{5}{2} = \binom{4}{2} + \binom{4}{1} \qquad \binom{5}{3} = \binom{4}{3} + \binom{4}{2} \qquad \binom{5}{4} = \binom{4}{4} + \binom{4}{3}$$