Current Time:

10:08 AM

Why Markov Analysis ?

Vito Faraci Jr.

Introduction (Personal Info)

$$P_f = \int_0^t pdf(x)dx$$

- Vito Faraci Jr.
- Mathematician by education, engineer by trade.
- Worked for small consulting company on L.I. for past 15 years presenting seminars to FAA on Probability, Reliability, FTA, FMEA, & MA.
- Have given MA lectures to Lockheed Martin System Safety Groups, Sandia National Labs, and to reliability engineers at Bombardiere Aircraft Co.
- Have several papers published by RIAC and System Safety Society on Reliability and MA.

Introduction (Background Info)

$$P_f = \int_0^t pdf(x)dx$$

Fault Tree Analysis (FTA) was introduced in 1962 at Bell Labs, and is the most commonly used tool for qualitative and quantitative risk analyses.

For a period of time its limitations were unquestioned but were known only to a few. Starting in the early 80s, a group of NASA mathematicians performed some impressive studies that clearly exposed a very subtle limitation. In an effort to overcome this limitation, NASA developed several algorithms, and described in detail an approach using Markov Analysis (MA) , designed not to replace, but to support FTAs.

With respect to Reliability , the integration of MA with FTA has been a giant step forward. Engineers can now solve more accurately a much larger set of "Risk" problems than they could before.

Introduction cont.

$$P_f = \int_0^t pdf(x)dx$$

MA was introduced in 1907 by a Russian mathematician by the name of A. A. Markov. It is interesting to note that although this knowledge has been around for some time, the engineering community had waited until the 1980s to taken advantage of this science. For example,

a) NASA has been employing Markov methods for Probabilistic Risk Assessments (PRA) for the Shuttle systems, and
b) FTA and Reliability Software manufacturers have integrated Markov techniques into their Risk Assessment SW Programs.

It is this author's opinion, that due to a lack of documentation written in a clear common language, knowledge of Markov as applied to Reliability still remains a little "sketchy" within the engineering community.

Acronyms

$$P_f = \int_0^t pdf(x)dx$$

- ARP – Aerospace Recommended Practice
- Combo - Combinatorial
- DE – Differential Equation
- FAA – Federal Aviation Administration
- FR – Failure Rate
- FTA – Fault Tree Analysis
- MA – Markov Analysis
- Non-combo - Non-combinatorial
- RBD – Reliability Block Diagram
- ROF – Required Order Factor
- SDE – Simultaneous Differential Equations
- SSD – State Sequence Diagram

Agenda

$$P_f = \int_0^t pdf(x)dx$$

Component Reliability vs. System Reliability

Component Rel vs. System Rel

$$P_f = \int_0^t pdf(x)dx$$

Reliability, sometimes referred to as Failure Analysis, is defined to be the probability that a component , black box, or system will perform its specified function successfully within a specified period of time.

The subject of Reliability can be divided into two major sub-categories, each of which is a science in itself requiring university studies to treat each of them adequately.

They are:

a) Component Reliability

and

b) System Reliability

Component Reliability

$$P_f = \int_0^t pdf(x)dx$$

Component Reliability is the science of measuring and calculating reliability of primary components such as transistors, resistors, capacitors, IC chips, ball bearings etc. It is the study of determining a component's failure characteristics based on various stresses that it will be exposed to. Stresses like hours of operation, temperature, humidity, vibration, voltage, current, etc. This science relies very heavily on **applied physics**. The final output product of this study is a mathematical model of the individual component's failure characteristic.

Failure characteristic of any primary component will fall into one of two categories either constant failure rate or non-constant failure rate.

A brief description of both categories is presented next.

Component Reliability cont.

Constant Failure Rate

$$P_f = \int_0^t pdf(x)dx$$

Failure characteristic of a constant failure rate device

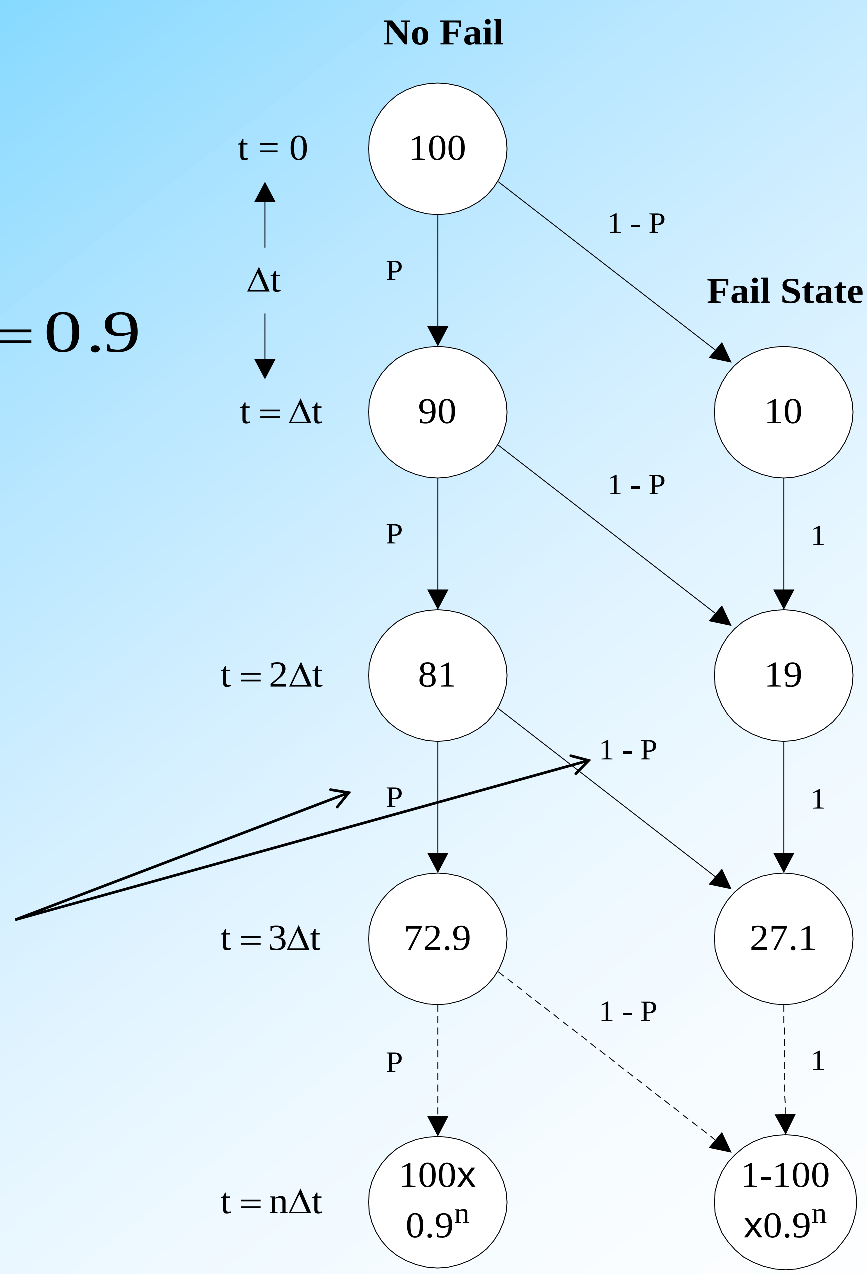
probability of success = $P = e^{-\lambda t} = 0.9$

probability of fail = $F = 1 - P$

where e = 2.71828,

λ = constant failure rate,

t = time.



Transition Probabilities

Note:
Percentage of failures are always the same (constant) for each time interval Δt.

Component Reliability cont.

Non-Constant Failure Rate

$$P_f = \int_0^t pdf(x)\,dx$$

The following is an example of a probability of failure (P_f) equation of a non-constant failure rate device. In this case, the device exhibits a "normal" distribution of failure:

$$P_f = \frac{1}{s\sqrt{2\pi}} \int_0^t e^{-\left(\frac{(x-u+hl)^2}{2s^2}\right)} dx$$

where

u = mean time to failure,

s = standard deviation,

hl = hours previously logged,

t = time.

x = dummy variable

Note:

The above equation is "non-integrable" which presents an additional challenge when calculating P_f.

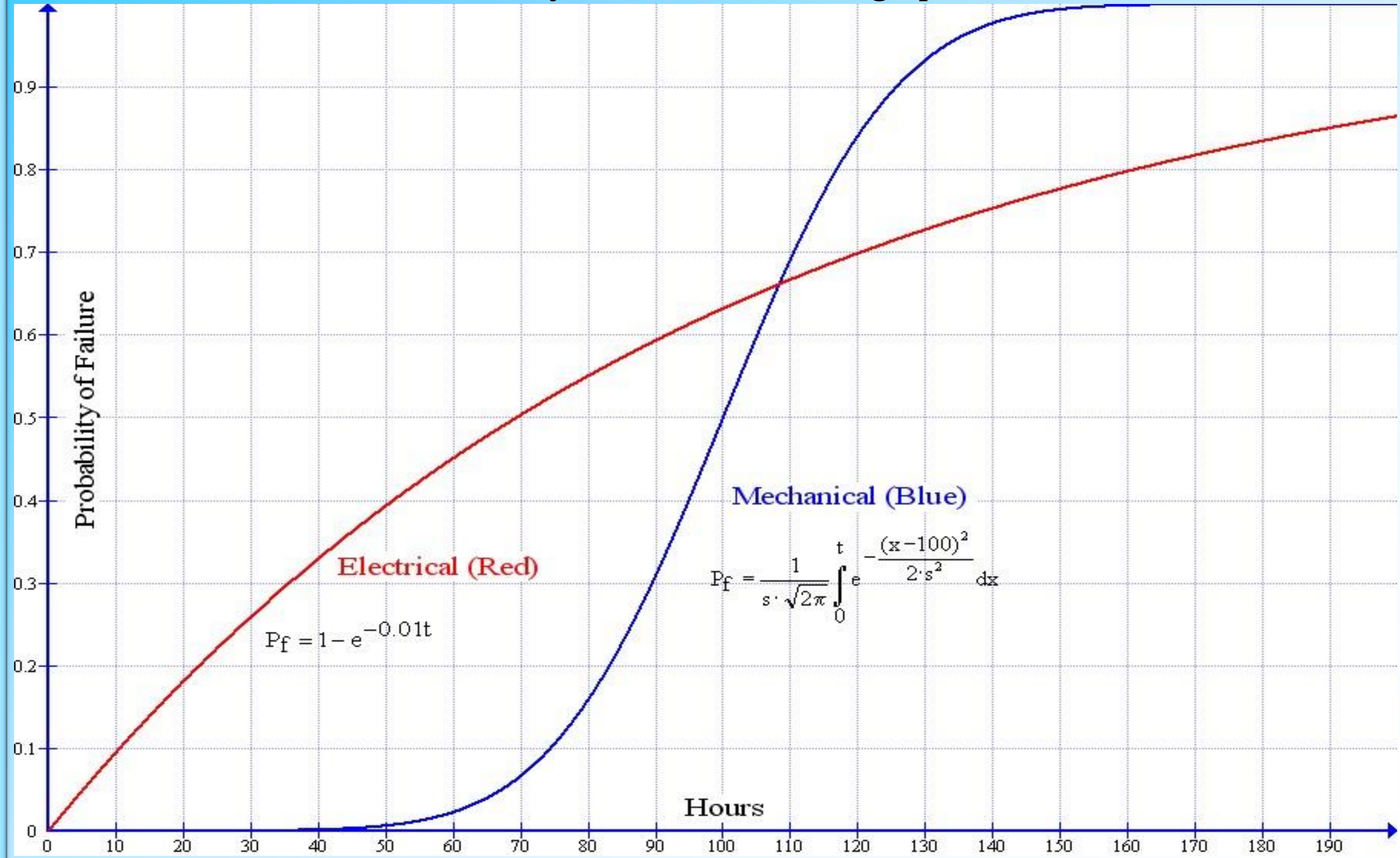
Component Reliability cont.

Constant FR vs. Non-Constant FR

$$P_f = \int_0^t pdf(x)dx$$

Comparison of a Constant and a Non-constant Failure Rate Device

Probability of failure vs. Time graph



System Reliability

$$P_f = \int_0^t pdf(x)dx$$

System Reliability is the science of measuring and/or calculating reliability of a system made up of two or more components. In other words creating a math model of the failure characteristic of the system. This is accomplished by deriving the failure characteristic (math model) of each individual component, and then **mathematically integrating individual component failure characteristics** based on the component inter-relationships operating in the system. This science obviously relies very heavily on applied mathematics.

The rest of this presentation is devoted to the study of this subject.

System Reliability cont.
Simple System

$$P_f = \int_0^t pdf(x)\,dx$$

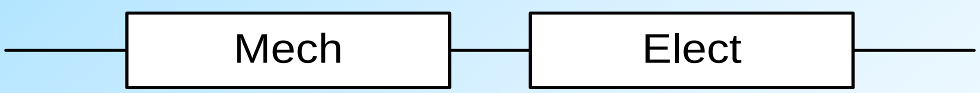
Mechanical device in **series** with an Electrical Device

$$P_{fm}(t) = P_f\,(\text{mech device}) \qquad P_{fe}(t) = P_f\,(\text{elect device})$$

$$P_f\,(\text{System}) = P_{fm}(t) + P_{fe}(t) - P_{fm}(t) \cdot P_{fe}(t)$$

$$P_{fm}(t) = \frac{1}{s \cdot \sqrt{2\pi}} \int_0^t e^{-\frac{(x-100)^2}{2 \cdot s^2}}\,dx, \qquad P_{fe}(t) = 1 - e^{-0.01t}$$

RBD



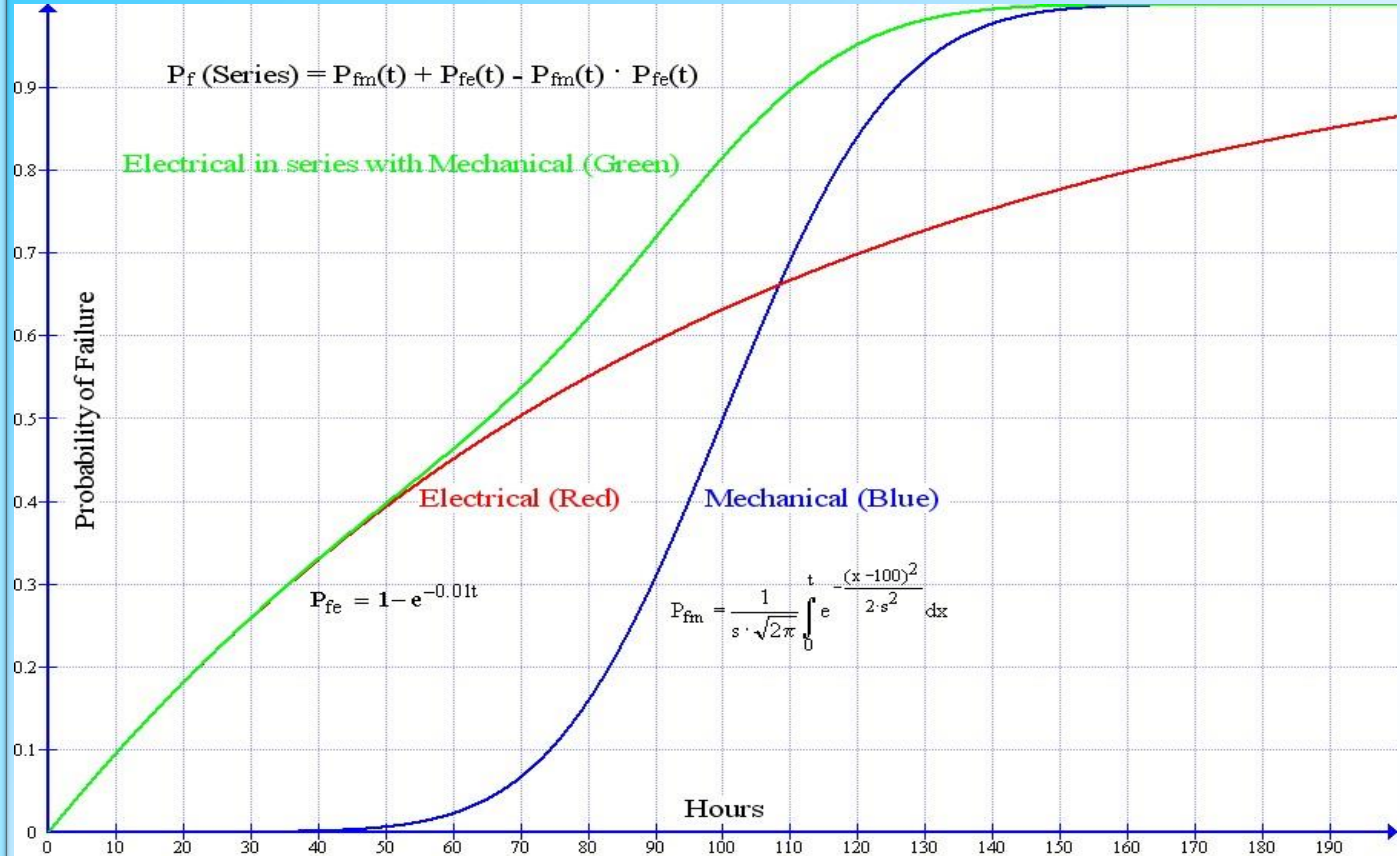
Major point of slide:
When integrating devices with different failure characteristics, all is required is basic "and/or" logic. No need for Markov or any other special techniques.

System Reliability cont.
Simple System

$$P_f = \int_0^t pdf(x)\,dx$$

Constant & Non-constant FR Devices in Series

Probability of failure vs. Time graph



$$P_{fm} = \frac{1}{s \cdot \sqrt{2\pi}} \int_0^t e^{-\frac{(x-100)^2}{2 \cdot s^2}} dx$$

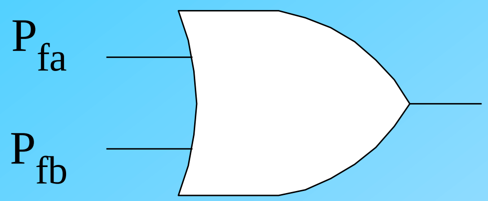
Quick Review of FTA Basics

(Very quick. FTA is a subject all by itself.)

Quick Review of FTA Basics cont.

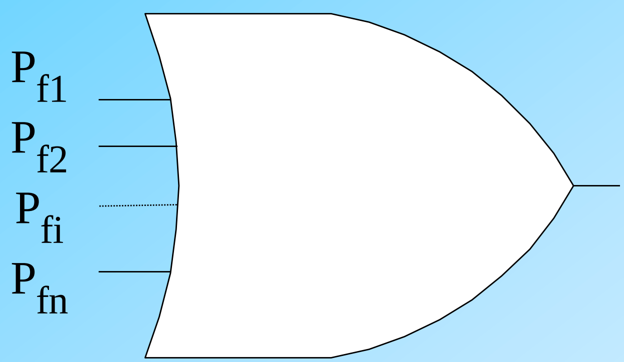
Or – Logic

$$P_f = \int_0^t pdf(x)dx$$



$$P_f = P_{fa} + P_{fb}$$

$$P_f = P_{fa} + P_{fb} - P_{fa} \cdot P_{fb}$$
(if not mutually exclusive)



$$P_f = P_{f1} + P_{f2} + \cdots + P_{fn}$$

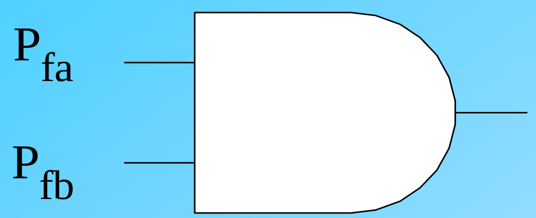
RBD – Series Configuration

Assuming all probabilities of failure mutually exclusive, then $P_f = \sum_{i=1}^{n} P_{f\,i}$

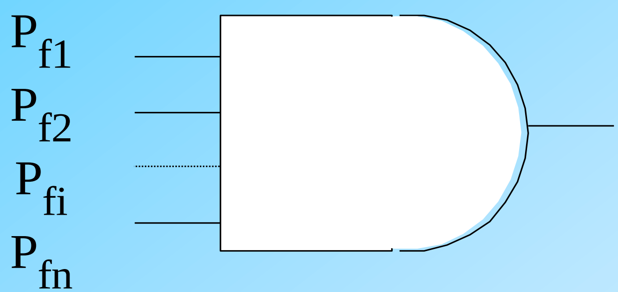
Quick Review of FTA Basics cont.

And – Logic

$$P_f = \int_0^t pdf(x)\,dx$$

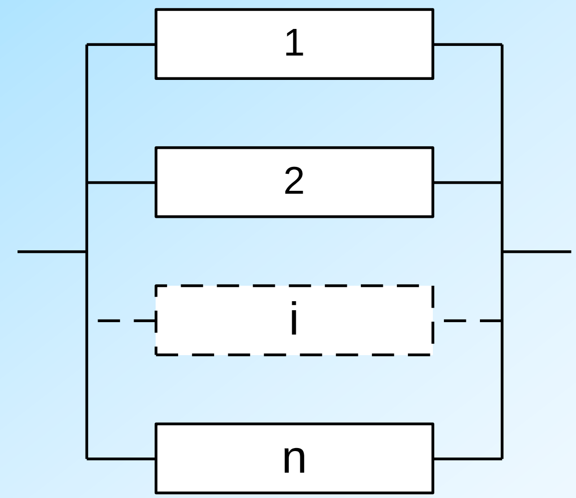


$$P_f = P_{fa} \cdot P_{fb}$$



$$P_f = P_{f1} \cdot P_{f2} \cdots P_{fn}$$

RBD – Parallel Configuration



Assuming all probabilities of failure independent, then $P_f = \prod_{i=1}^{n} P_{f\,i}$

Important notes:

1) FTA logic is not limited to exponentials as shown on a previous slide.
2) Not all reliability problems can be reduced to series or parallel models (logic).

FTA vs. MA Advantages / Disadvantages

$$P_f = \int_0^t pdf(x)dx$$

FTA vs. MA Advantages / Disadvantages

FTA vs. MA Advantages / Disadvantages

$$P_f = \int_0^t pdf(x)dx$$

Fault Tree Advantages:

- Acts as a visual tool which can be used to pinpoint system weaknesses.
- Exhibits clear representation of logical processes that lead to a system or sub-system failure (clear qualitative representation of failure propagation).
- Reveals relatively simple equations for P_f calculations yielding quantitative analyses that do not require high powered math.
- Proves to be a very effective tool for the fault isolation process.

FTA vs. MA Advantages / Disadvantages cont.

$$P_f = \int_0^t pdf(x)dx$$

Fault Tree Limitations:

- Difficult to allow for transient & intermittent faults or standby systems with spares.
- If a system has many failure conditions, separate fault trees may need to be constructed for each one.
- Difficult to represent systems where failure rates or repair rates are state dependent (change between states).

From ARP 4761 Issue 1996-12

FTA vs. MA Advantages / Disadvantages cont.

$$P_f = \int_0^t pdf(x)dx$$

The following is an excerpt from NASA Ref. Publication 1348:

Traditionally, the reliability analysis of a complex system has been accomplished with combinatorial mathematics. The standard fault-tree method of reliability analysis is based on such mathematics. Unfortunately, the fault-tree approach is somewhat limited and incapable of analyzing systems in which reconfiguration is possible. Basically, a fault tree can be used to model a system with:

Only permanent faults (no transient or intermittent faults)

- No reconfiguration
- No time or sequence failure dependencies
- No state-dependent behavior (e.g., state-dependent failure rates)

FTA vs. MA Advantages / Disadvantages cont.

$$P_f = \int_0^t pdf(x)dx$$

- MA does not have these limitations.
- Sequence dependent events are included and handled naturally.
- Covers a much wider range of system behaviors.

From ARP4761 Issue 1996-12

Close examination of the above NASA and FAA excerpts reveals the answer to the "Why Markov" question. It has to do with combinatorial vs. non-combinatorial logic.

FTA vs. MA Advantages / Disadvantages cont.

$$P_f = \int_0^t pdf(x)dx$$

- **Combinatorial Logic**

  Two or more input states define one or more output states. Output states are related by defined rules that are independent of previous states.

  - Logic depends solely on combinations of inputs
  - Time is neither modeled or recognized
  - Outputs change when inputs change irrespective of time
  - Output is a function of, and only of, the present input
  - Logic can be represented using And Gates & Or Gates (Fault Tree)

- **Non-combinatorial Logic (Example: Sequential Logic)**

  Logic of output(s) depends on combinations of present input states, and combinations of previous input states

In other words non-combinatorial logic has memory while combinatorial logic does not.

FTA vs. MA Advantages / Disadvantages cont.

$$P_f = \int_0^t pdf(x)dx$$

- Fault Tree Analysis (FTA) –Handles combinatorial type problems both qualitatively and quantitatively extremely well. However FTA has difficulty with non-combinatorial problems in both areas.
- Markov Analysis (MA) – Handles non-combinatorial as well as combinatorial problems. However, not quite as intuitive as FTA, and usually requires higher power mathematics for quantitative analyses.
- **Why Markov?**

  Stated mathematically Markov handles non-combinatorial problems that FTA cannot.

Markov Analysis Basics

Markov Analysis Basics

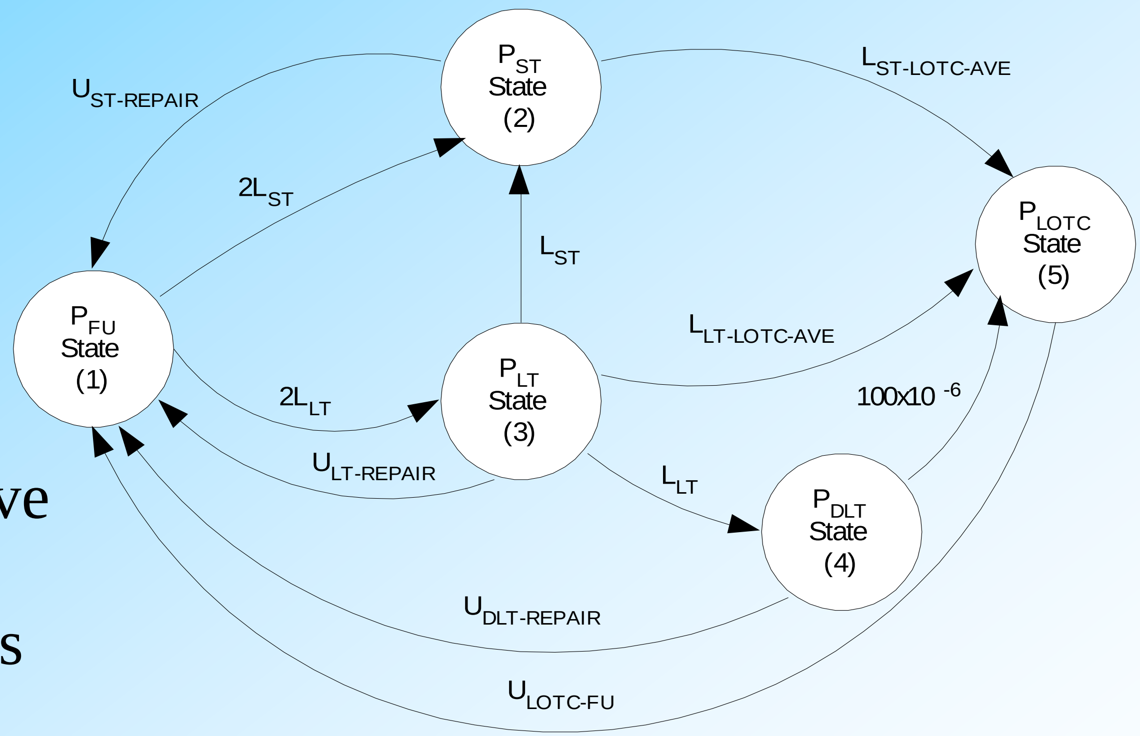
$$P_f = \int_0^t pdf(x)dx$$

If a system or component can be in one of two states (i.e. failed, non-failed), and if we can define the probabilities associated with these states on a discrete or continuous basis, the probability of being in one or other at a future time can be evaluated using a state-time analysis. In reliability and availability analysis, failure probability and the probability of being returned to an available state are the variables of interest. The best known state-space technique is Markov Analysis.

Markov Analysis Basics cont.

$$P_f = \int_0^t pdf(x)dx$$

- State Diagram represents various system states
- Transition rate is function of failure or repair rate
- States must be finite in number
- States are mutually exclusive
- The sum of the probabilities must equal 1



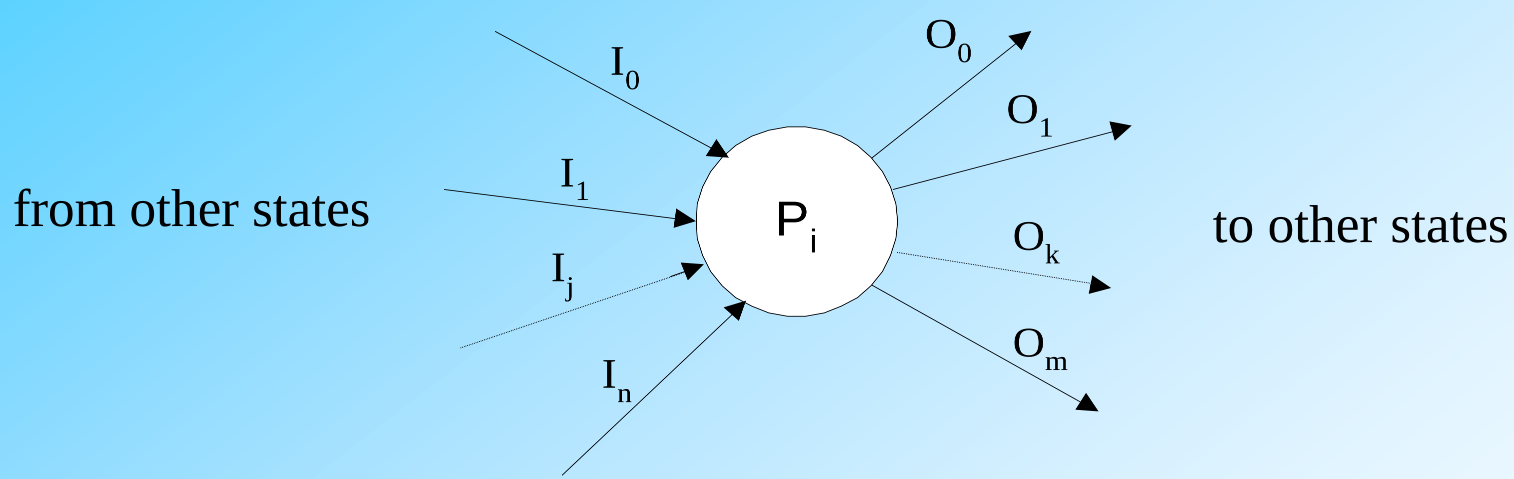
Mutually Exclusive – System can never be in any more than one state at any given time

Markov Analysis Basics cont.

Math Modeling (Determining SDE)

$$P_f = \int_0^t pdf(x)dx$$

The following is a typical state taken from a State Diagram with n input transitions with constant failure rates I_j, and m output transitions with constant failure rates O_k.



Pi = probability of being in state i

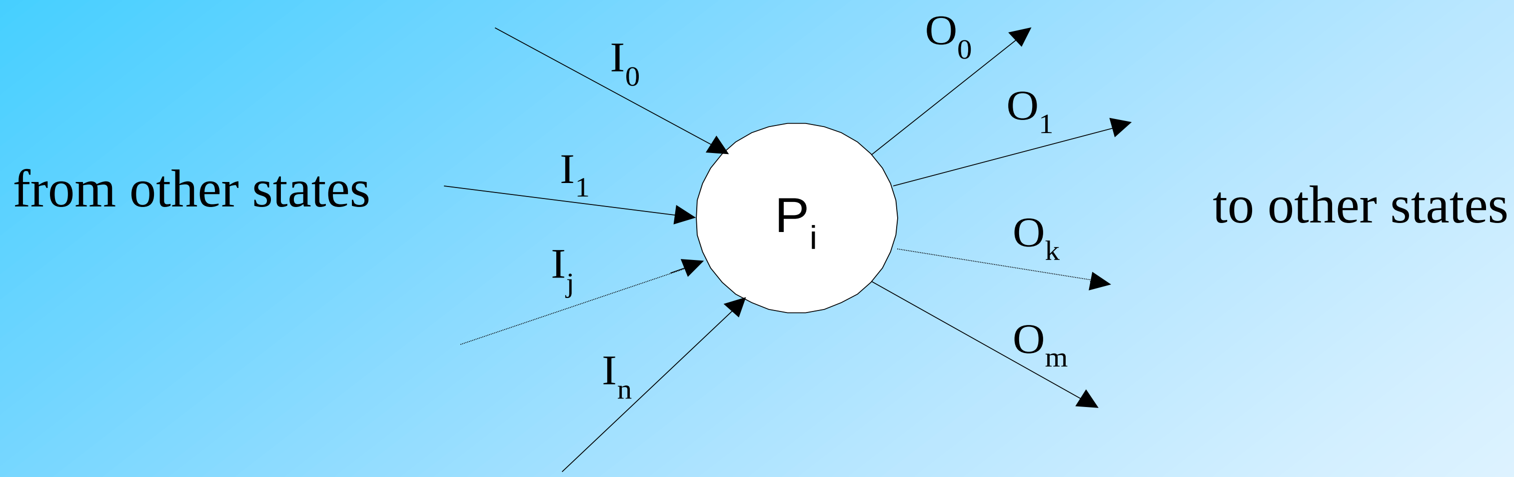
Unfortunately Pi cannot be calculated immediately. Calculation of Pi requires the solution of a set of simultaneous differential equations (SDE).

However, determination of the SDE is really quite simple once an accurate State Diagram of the system or sub-system is constructed.

Markov Analysis Basics cont.

Math Modeling (Determining SDE) cont.

$$P_f = \int_0^t pdf(x)dx$$



Each state has a corresponding DE. The DE corresponding to a typical state is:

$$\frac{dP(i)}{dt} = \sum_{j=1}^{n} I_j P(j) - \left(\sum_{k=1}^{m} O_k \right) P(i)$$

Note: Transitions <u>into</u> a state result in <u>positive</u> terms in the DE, while transitions <u>leaving</u> a state yield <u>negative</u> terms.

For the sake of simplifying notation let $P_i = P(i)$.

Markov Analysis Basics cont.

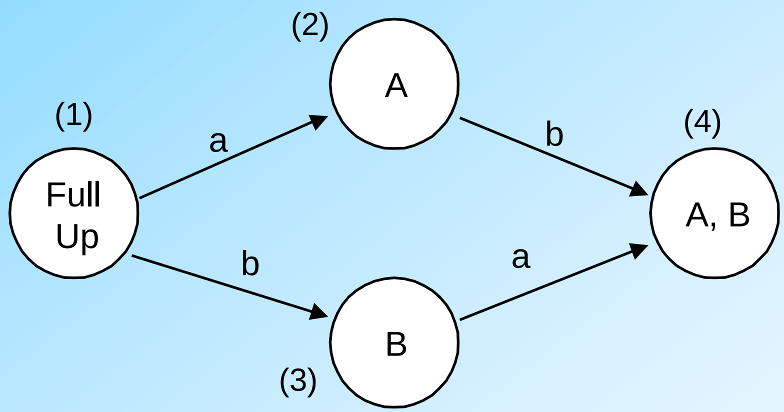
Math Modeling (Determining SDE)

$$P_f = \int_0^t pdf(x)dx$$

Example : Two devices in Parallel

Transitions **into** a state result in **positive** terms in the DE, while transitions **leaving** a state yield **negative** terms.

$$\frac{dP_2}{dt} = aP_1 - bP_2$$



$$\frac{dP_1}{dt} = -(a+b)P_1$$

$$\frac{dP_4}{dt} = bP_2 + aP_3$$

$$\frac{dP_3}{dt} = bP_1 - aP_3$$

$$P_f = \int_0^t pdf(x)dx$$

Markov Analysis Compared with FTA

Markov Analysis Compared with FTA

$$P_f = \int_0^t pdf(x)dx$$

Combinatorial

Type

Problems

Markov Analysis Compared with FTA

Combinatorial Type Problems

$$P_f = \int_0^t pdf(x)dx$$

As mentioned before Markov handles combinatorial as well as non-combinatorial problems.

Although there is no need of Markov for solving combinatorial type problems, (FTA handles them well enough) the next few slides will demonstrate several examples for the sake of illustration and comparison.

Note:

The following comparison examples are limited to "constant failure rate" type problems. Solutions to "non-constant failure rate" type problems require different math techniques.

Markov Analysis Compared with FTA

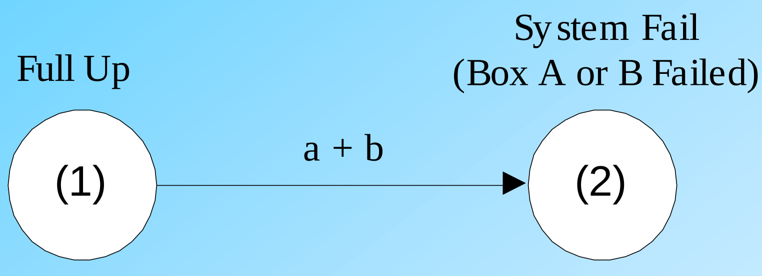
2 Components in Series (Combo Type)

$$P_f = \int_0^t pdf(x)dx$$

Two black boxes start operation at the same time. Box 1 has failure rate a and Box 2 has failure rate b. Successful system operation requires that both boxes be working.

Find P_f = Probability of System Failure.

State Diagram

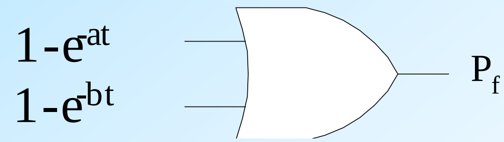


$$dP(2)/dt = (a+b)P(1) \Rightarrow$$

$$P_f = P(2) = 1 - e^{-(a+b)t}$$

FTA Approach

(and/or logic)



$$P_f = P_a + P_b - P_a \cdot P_b \Rightarrow$$

$$P_f = 1 - e^{-(a+b)t}$$

Note: P(n) = Probability of State (n)

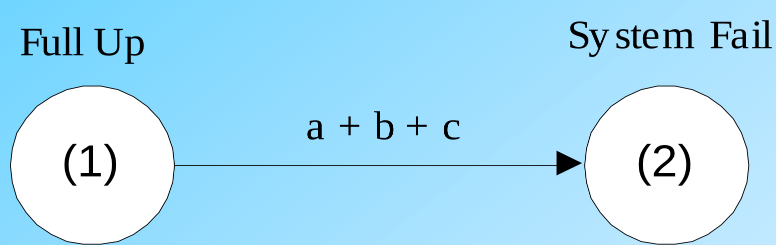
Markov Analysis Compared with FTA

3 Components in Series (Combo Type)

$$P_f = \int_0^t pdf(x)dx$$

Three black boxes start operation at the same time. Boxes 1, 2, and 3 have failure rates a, b, and c respectively. Successful system operation requires that all three boxes be working. Find P_f the Probability of System Failure.

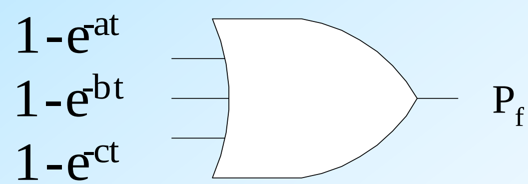
State Diagram



$$dP(2)/dt = (a+b+c)\cdot P(1)$$

$$\Rightarrow P_f = P(2) = 1 - e^{-(a+b+c)t}$$

FTA Approach
(and/or logic)



$$P_f = P_a + P_b + P_c - P_a \cdot P_b - P_a \cdot P_c - P_b \cdot P_c + P_a \cdot P_b \cdot P_c$$

$$\Rightarrow P_f = 1 - e^{-(a+b+c)t}$$

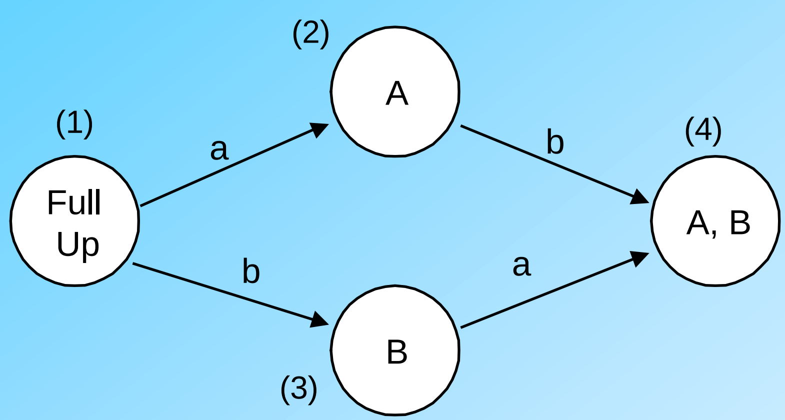
Markov Analysis Compared with FTA

2 Components Active Redundant (Parallel) (Combo Type)

$$P_f = \int_0^t pdf(x)dx$$

Two black boxes start operation at the same time. Box 1 has failure rate a and Box 2 has failure rate b. Successful system operation requires that Box 1 or Box 2 or both be working. Find P_f the Probability of System Failure.

State Diagram

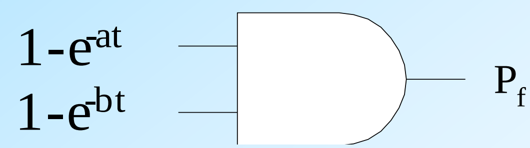


$$dP(4)/dt = bP(2) + aP(3)$$

$$\Rightarrow P_f = P(4) = (1 - e^{-at})(1 - e^{-bt})$$

FTA Approach

(and/or logic)



$$P_f = (1 - e^{-at})(1 - e^{-bt})$$

Note:

Failure rates are not effected by state changes.

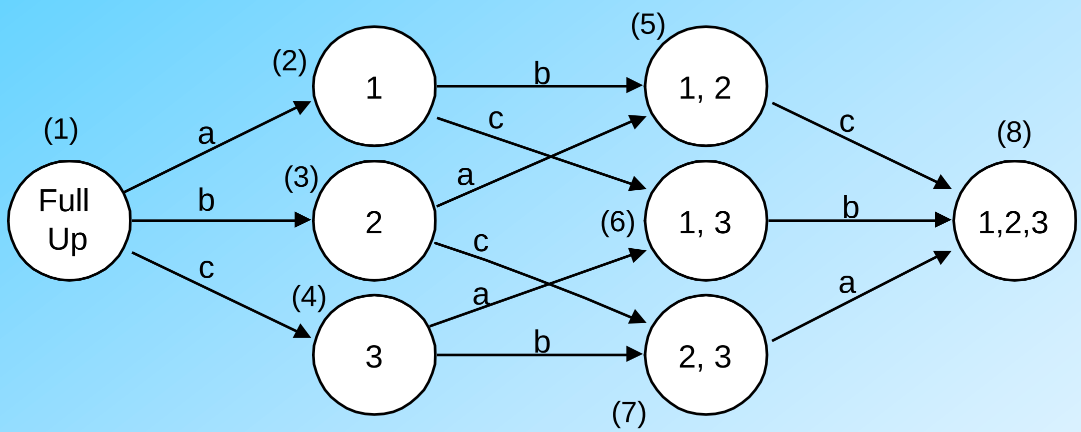
Markov Analysis Compared with FTA

3 Components Active Redundant (Parallel) (Combo Type)

$$P_f = \int_0^t pdf(x)dx$$

Three black boxes start operation at the same time. Box 1, 2, and 3 have failure rate a, b, and c respectively. Successful system operation requires that Box 1, 2, or 3 be working. Find P_f the Probability of System Failure.

State Diagram

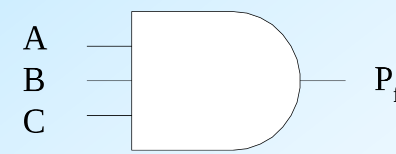


$$dP(8)/dt = cP(5) + bP(6) + aP(7)$$

$$\Rightarrow P_f = P(8) = (1-e^{-at})(1-e^{-bt})(1-e^{-ct})$$

FTA Approach

(and/or logic)



$$A = (1-e^{-at})$$

$$B = (1-e^{-bt})$$

$$C = (1-e^{-ct})$$

$$P_f = (1-e^{-at})(1-e^{-bt})(1-e^{-ct})$$

Markov Analysis Compared with FTA cont.

$$P_f = \int_0^t pdf(x)dx$$

Non-combinatorial

Type

Problems

Markov Analysis Compared with FTA

Non-combinatorial Type Problems

Solutions of non-combinatorial problems require different techniques other than traditional combinatorial logic such as that found in FTAs. In particular one of the simplest non-combinatorial type problem that has intrigued mathematicians is the classic “Standby Problem”.

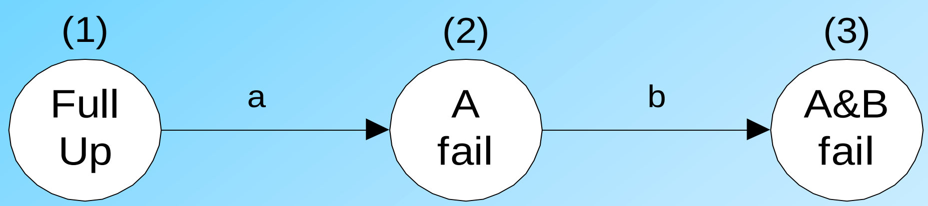
Markov Analysis Compared with FTA

Standby Problem (Non-Combo Type)

$$P_f = \int_0^t pdf(x)dx$$

Box A has failure rate a and Box B has failure rate b. Box A is turned on while Box B remains powered off in standby mode. Immediately upon detection of Box A failure, Box B is turned on. Calculate the probability that <u>both</u> boxes are failed.

State Diagram



FTA

?

$$\frac{dP_1}{dt} = -aP_1 \Rightarrow \qquad P_1 = e^{-at}$$

$$\frac{dP_2}{dt} = aP_1 - bP_2 \Rightarrow \quad P_2 = \frac{a}{a-b}\left(e^{-bt} - e^{-at}\right)$$

$$\frac{dP_3}{dt} = bP_3 \Rightarrow \qquad P_3 = 1 + \frac{b}{a-b}e^{-at} - \frac{a}{a-b}e^{-bt}$$

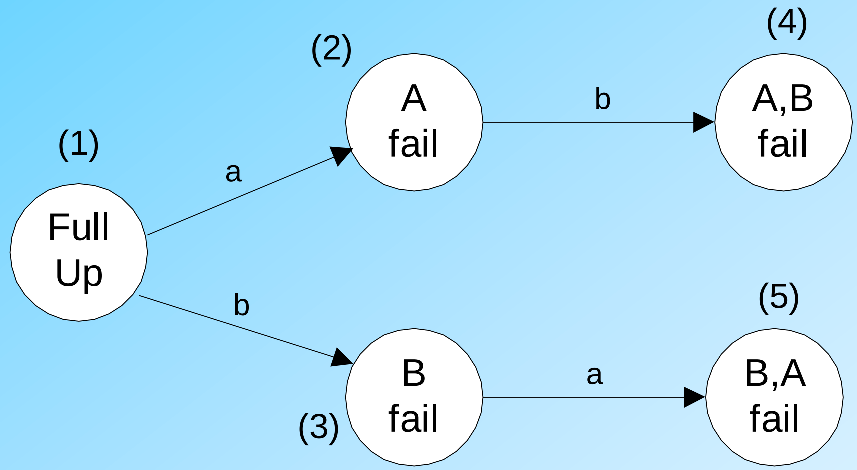
Markov Analysis Compared with FTA

2 Components Active Redundant with ROF (Non-Combo Type)

$$P_f = \int_0^t pdf(x)dx$$

Two components are in operation. Find the probability that both Boxes A and B fail and that Box A fails before Box B. Also find the probability that both Boxes fail and that Box B fails before Box A.

State Diagram



FTA

?

$$P(1) = e^{-at} \cdot e^{-bt}$$

$$P(2) = (1 - e^{-at})e^{-bt}$$

$$P(3) = (1 - e^{-bt})e^{-at}$$

$$P(4) = \frac{a}{a+b} + \frac{b}{a+b} e^{-(a+b)t} - e^{-bt}$$

$$P(5) = \frac{b}{a+b} + \frac{a}{a+b} e^{-(a+b)t} - e^{-at}$$

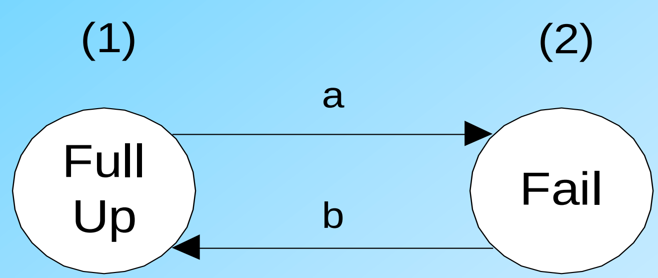
Markov Analysis Compared with FTA

Component with Repair (Non-Combo Type)

$$P_f = \int_0^t pdf(x)dx$$

A Black Box has failure rate a and a repair rate b. Upon detection of a failure, the Box goes into a repair process and put back on line. Calculate the probability that the Box will be available.

State Diagram



$$P(1) = \frac{b}{a+b} + \frac{a}{a+b} e^{-(a+b)t}$$

$$P(2) = \frac{a}{a+b} - \frac{a}{a+b} e^{-(a+b)t}$$

FTA

?

Past Attempt to Modify FTA

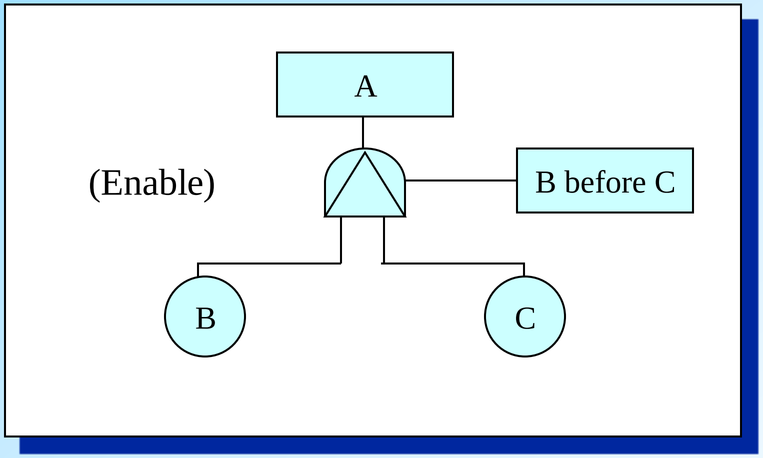
$$P_f = \int_0^t pdf(x)dx$$

Past Attempt to Modify FTA to Handle Non-combinatorial Type Problems

FTA Required Order Factors

$$P_f = \int_0^t pdf(x)dx$$

- Establish any relevant required order factors (ROF)
  - An AND-gate in a fault tree implies no specific order of the faults present. In some cases, this may be unrealistic.
    - An example is a failure combination where a monitor is used to detect failures of functional circuitry that can cause the top level event.
    - If the monitor fails first, the failure may remain latent until the monitor is checked.
    - If the function c circuitry fails first, the top level event does not occur because the monitor annunciates the failure.



Event A will occur only if event B occurs and subsequently event C occurs.

From ARP 4761 Issue 1996-12

FTA Required Order Factors cont.

$$P_f = \int_0^t pdf(x)dx$$

Rule of Thumb:

- When dealing with failure order dependent events, a factor may be incorporated into the fault tree to make the calculated probabilities less conservative. This factor is known as the Required Order Factor (ROF) or the Sequencing Factor.
- For $\lambda t < 0.1$ the probability of the two events occurring in either order (given that they both fail) is approximately ½ of the total probability and therefore the ROF for each order is ½.
- In general, if there are n events in an AND-gate there are n! possible orders in which they could fail. If only k of those possible orders lead to the top event, then ROF = k/n!
- This approximation is only valid for events with the same exposure time or events with different exposure times where $(\lambda_1 + \lambda_2)\ T_{(Max)}$ is less than 0.2.

  For all other cases, ROF should be calculated.

From ARP 4761 Issue 1996-12

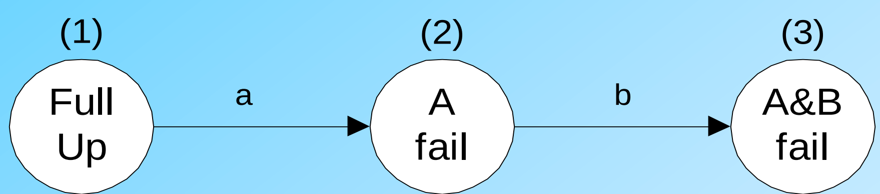
Markov Analysis Compared with FTA

Standby Problem (Non-Combo Type)

$$P_f = \int_0^t pdf(x)dx$$

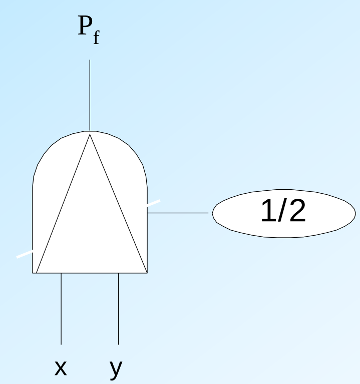
Box A has failure rate a and Box B has failure rate b. Box A is turned on while Box B remains powered off in standby mode. Immediately upon detection of Box A failure, Box B is turned on. Calculate the probability that <u>both</u> boxes are failed.

State Diagram



$$P(3) = 1 + \frac{b}{a-b}e^{-at} - \frac{a}{a-b}e^{-bt}$$

FTA using ROF



$$x = (1 - e^{-at}), \quad y = (1 - e^{-bt})$$

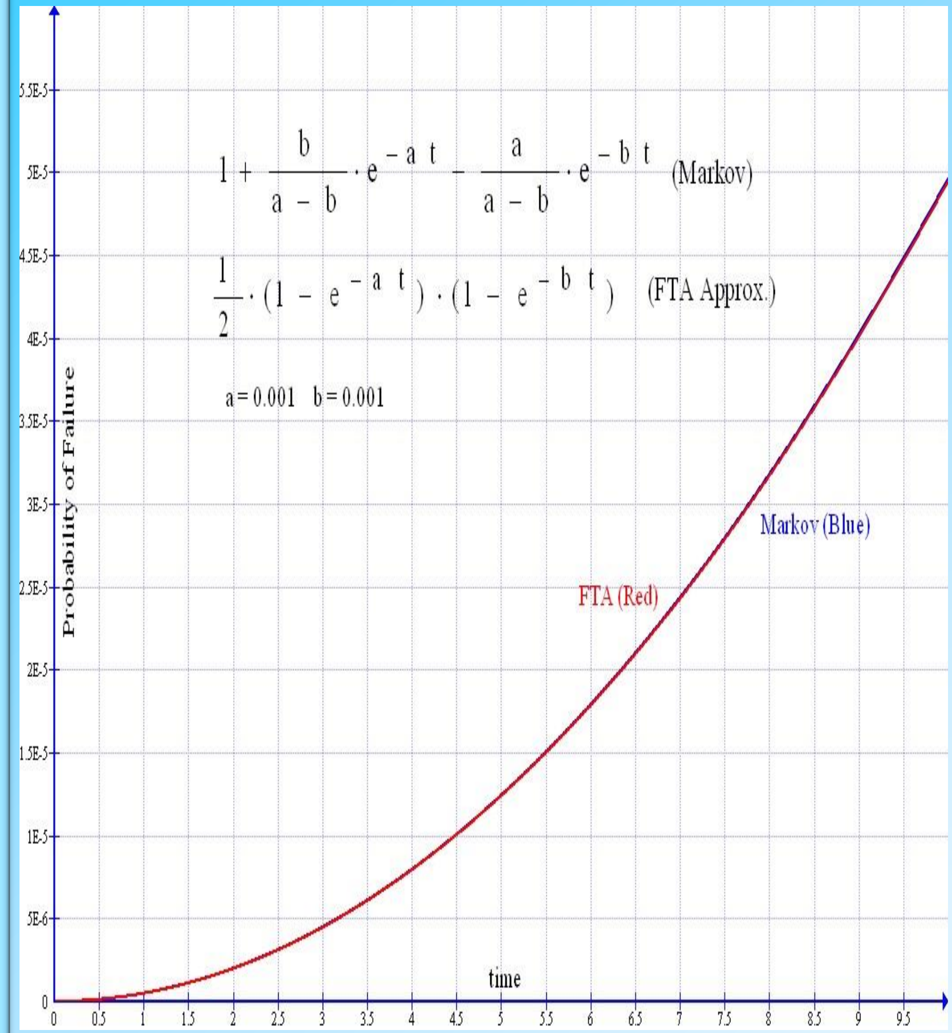
$$P_f = ½xy = ½(1 - e^{-at})(1 - e^{-bt})$$

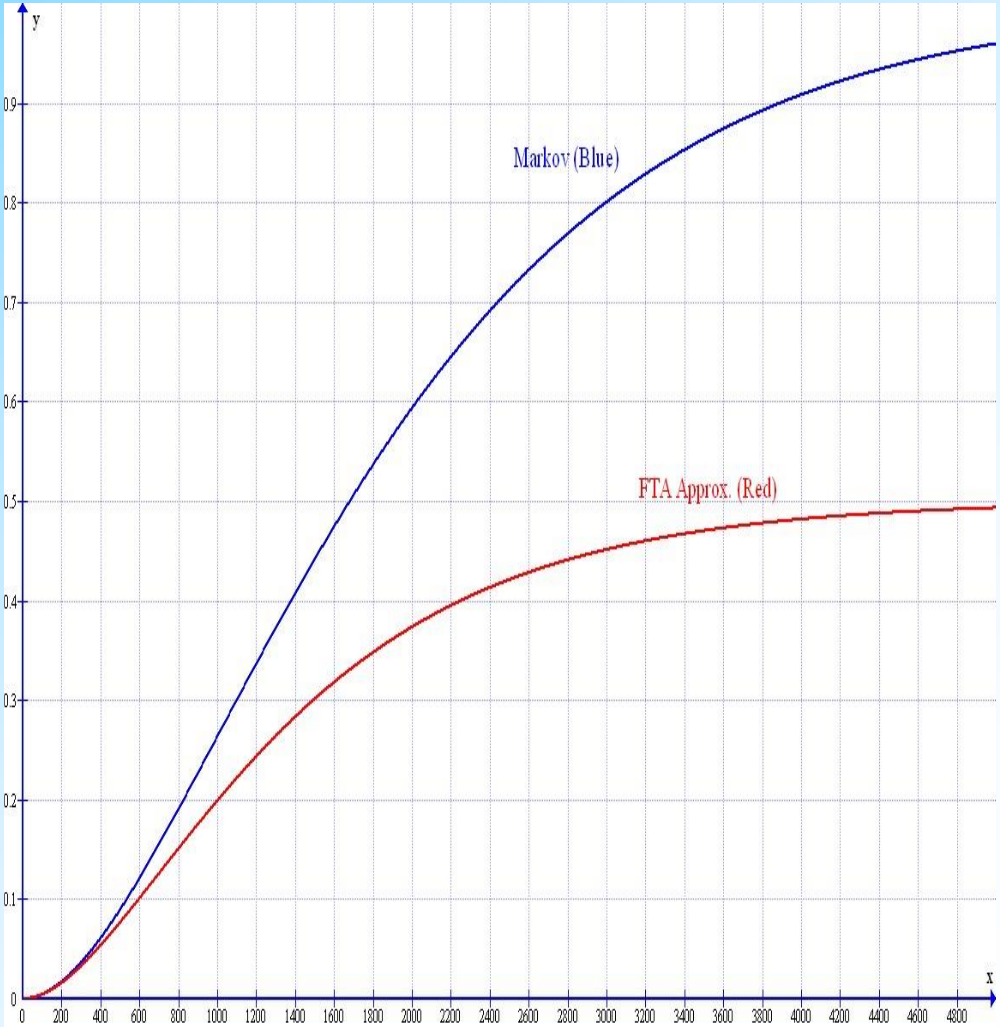
Markov Analysis Compared with FTA

Graph of Standby Problem (Non-Combo Type)

$$P_f = \int_0^t pdf(x)dx$$

Pf vs. Time (0 to 10 hours) (0 to 5000 hours)





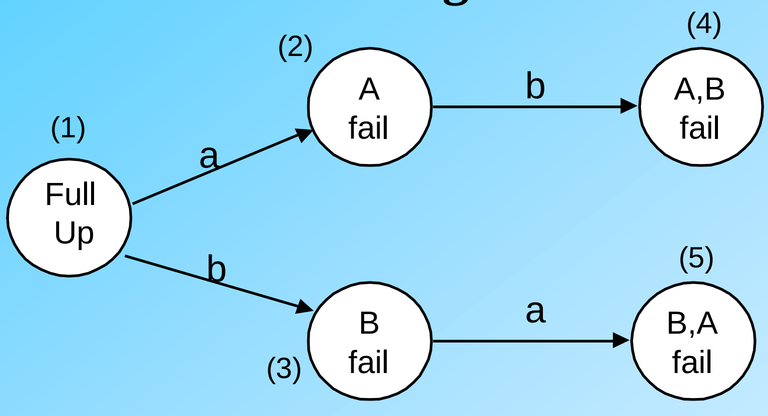
Markov Analysis Compared with FTA

2 Components Active Redundant with ROF (Non-Combo Type)

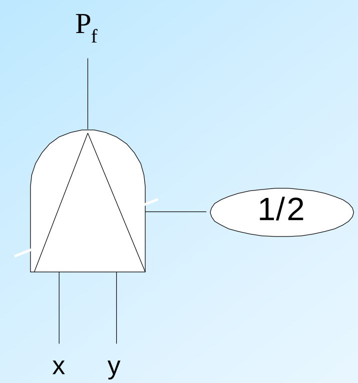
$$P_f = \int_0^t pdf(x)dx$$

Two components are in operation. Find the probability that both Boxes A and B fail and that Box A fails before Box B. Also find the probability that both Boxes fail and that Box B fails before Box A.

State Diagram



FTA using ROF



$$P(4) = \frac{a}{a+b} + \frac{b}{a+b} e^{-(a+b)t} - e^{-bt}$$

$$P(5) = \frac{b}{a+b} + \frac{a}{a+b} e^{-(a+b)t} - e^{-at}$$

$$x = (1 - e^{-at}), \; y = (1 - e^{-bt})$$

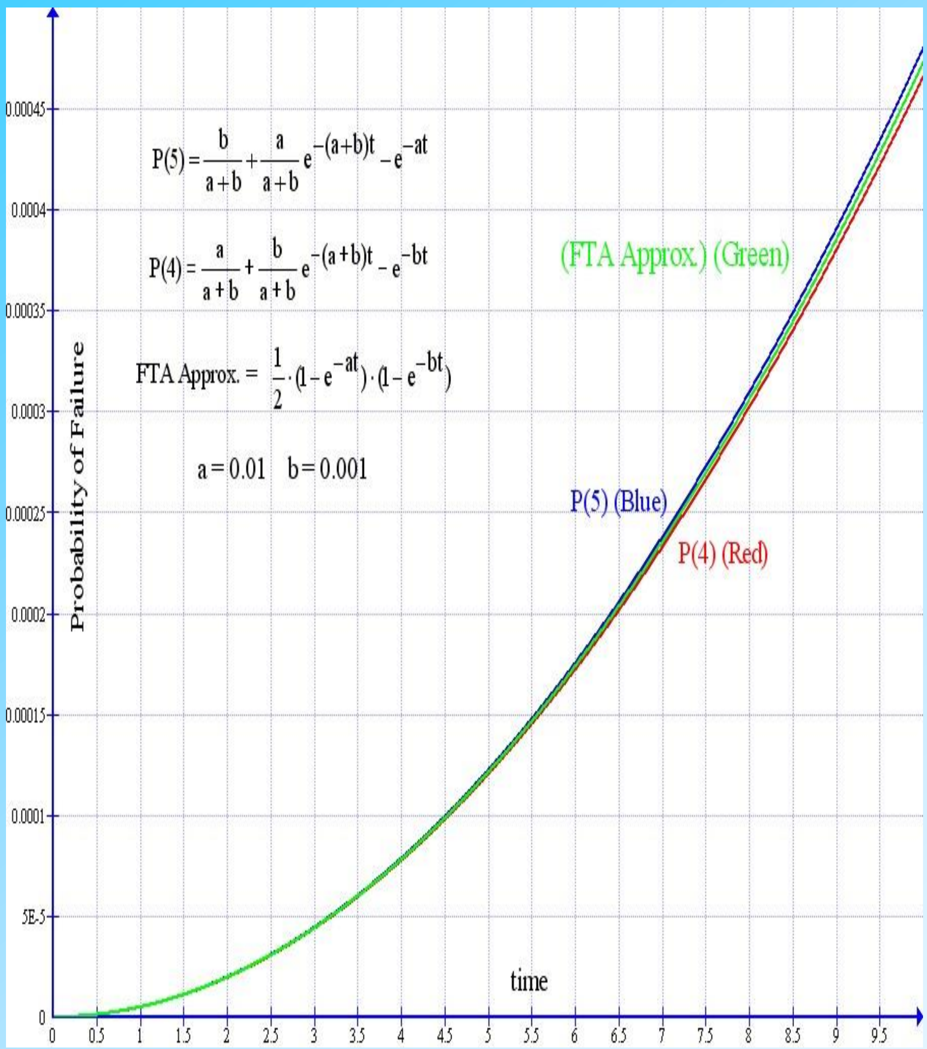
$$P_f = ½xy = ½(1 - e^{-at})(1 - e^{-bt})$$

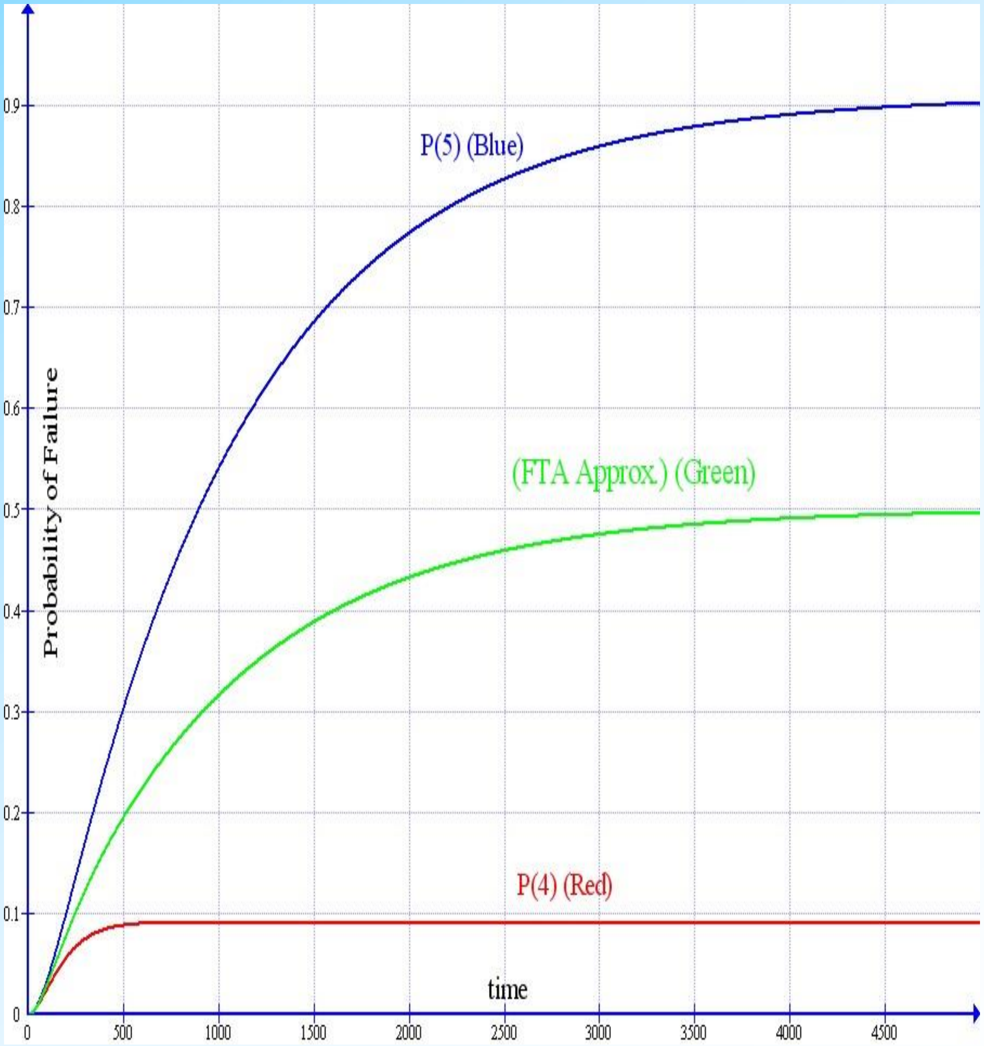
Markov Analysis Compared with FTA

Graph of ROF Problem (Non-Combo Type)

$$P_f = \int_0^t pdf(x)dx$$

Pf vs. Time (0 to 10 hours) (0 to 5000 hours)





Markov Analysis Compared with FTA

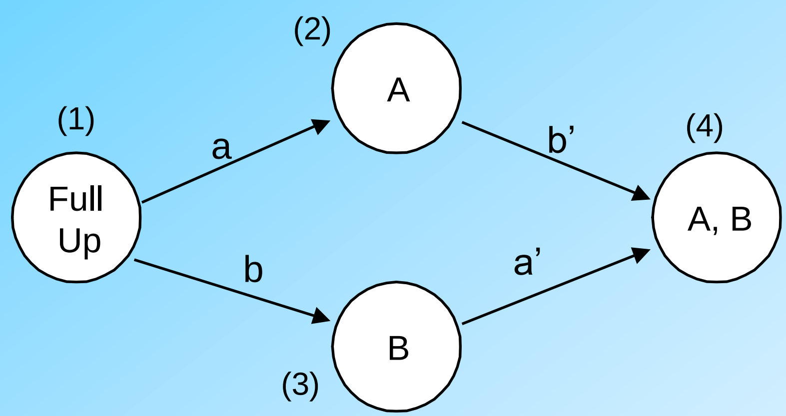
2 Components Active Redundant (Parallel) (Non-Combo Type)

$$P_f = \int_0^t pdf(x)dx$$

Two black boxes start operation at the same time. Box A has initial failure rate a, and then has failure rate a' when Box B fails due to increase stress. Box B has initial failure rate b, and then has failure rate b' when Box A fails. Successful system operation requires that Box A or Box B or both be working.

Find P_f the Probability of System Failure.

State Diagram



Equations

$$P(1) = e^{-(a+b)t}$$

$$P(2) = \frac{a}{a+b-b'}\left(e^{-b't} - e^{-(a+b)t}\right)$$

$$P(3) = \frac{b}{a+b-a'}\left(e^{-a't} - e^{-(a+b)t}\right)$$

$$P(4) = 1-[P(1)+P(2)+P(3)]$$

Important note:

Whenever failure rates change from one state to another, the problem becomes non-combinatorial.

Markov Analysis Compared with FTA

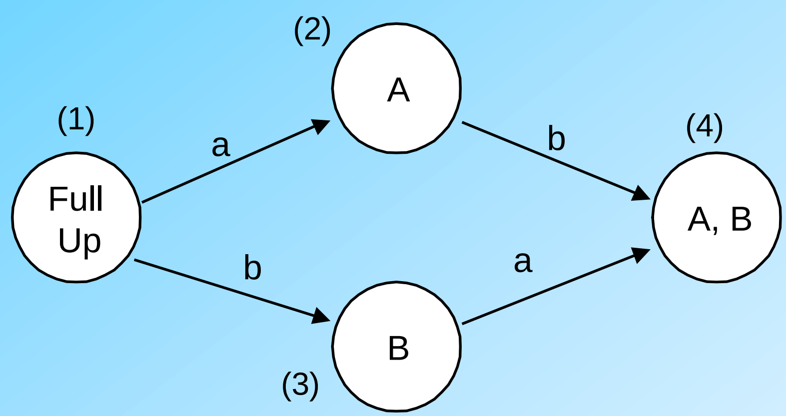
2 Components Active Redundant (Parallel) (Combo Type)

$$P_f = \int_0^t pdf(x)dx$$

Two black boxes start operation at the same time. Box A has failure rate a, and Box B has initial failure rate b. Successful system operation requires that Box A or Box B or both be working.

Find P_f the Probability of System Failure.

State Diagram



Equations

$$P(1) = e^{-at} \cdot e^{-bt}$$

$$P(2) = e^{-bt} \cdot (1 - e^{-at})$$

$$P(3) = e^{-at} \cdot (1 - e^{-bt})$$

$$P(4) = (1 - e^{-at}) \cdot (1 - e^{-bt})$$

Notes:

1) Notice the tell-tale characteristic of combinatorial equations in the solution set.
2) Compare this slide to the previous one where failure rates changed.

Applied Reliability Symposium, North America 2010

Review of Various Methods of Solution of SDEs

Solution to "Standby" using Laplace

$$P_f = \int_0^t pdf(x)dx$$

What follows is a method using Laplace Transforms for solving for P_1, P_2, and P_3 based on the 3 SDEs obtained from the Markov Diagram:

$$\frac{dP_1}{dt} = -aP_1, \quad \frac{dP_2}{dt} = aP_1 - bP_2, \quad \frac{dP_3}{dt} = bP_2 \Rightarrow$$

$$L\left(\frac{dP_1}{dt}\right) = L(-aP_1) \Rightarrow sL(P_1) - P_1(0) = -aL(P_1) \Rightarrow sL(P_1) - 1 = -aL(P_1) \qquad (1)$$

$$L\left(\frac{dP_2}{dt}\right) = L(aP_1 - bP_2) \Rightarrow sL(P_2) - P_2(0) = aL(P_1) - bL(P_2) \Rightarrow$$

$$sL(P_2) = aL(P_1) - bL(P_2) \qquad (2)$$

$$L\left(\frac{dP_3}{dt}\right) = L(bP_2) \Rightarrow sL(P_3) - P_3(0) = bL(P_2) \Rightarrow sL(P_3) = bL(P_2) \qquad (3)$$

$$(1) \Rightarrow L(P_1) = \frac{1}{s+a} \quad \text{and} (1) \& (2) \Rightarrow L(P_2) = \frac{a}{(s+a)(s+b)} \qquad (4)$$

Note: $P_1(0) = 1$ and $P_2(0) = 0$ assumed.

Solution to "Standby" using Laplace cont.

$$P_f = \int_0^t pdf(x)dx$$

$$(4) \Rightarrow P_1 = L^{-1}\left(\frac{1}{s+a}\right) = e^{-at} \text{ and } P_2 = L^{-1}\left(\frac{a}{(s+a)(s+b)}\right)$$

Using techniques from Partial Fractions $\frac{a}{(s+a)(s+b)} = \frac{a/(a-b)}{s+b} - \frac{a/(a-b)}{s+a} \Rightarrow$

$$P_2 = L^{-1}\left(\frac{a/(a-b)}{s+b}\right) - L^{-1}\left(\frac{a/(a-b)}{s+a}\right) \Rightarrow P_2 = \frac{a}{a-b}e^{-bt} - \frac{a}{a-b}e^{-at}$$

Note: The third DE in Line (3) could be used to solve for P3. However since P_1 and P_2 are known, use the fact that $P_1 + P_2 + P_3 = 1$. This approach is faster and simpler.

$$P_1 + P_2 + P_3 = 1 \Rightarrow P_3 = 1 - e^{-at} + \frac{a}{a-b}e^{-at} - \frac{a}{a-b}e^{-bt} \Rightarrow$$

$$\boxed{P_3 = 1 + \frac{b}{a-b}e^{-at} - \frac{a}{a-b}e^{-bt}}$$

Solution to "Standby" using a Formula

$$P_f = \int_0^t pdf(x)dx$$

Many Markov (SDE) problems can be solved using the following formula:

If $f(t)$ and $g(t)$ are functions of t, and $\frac{dP_i}{dt} = g(t) - f(t)P_i$

then $P_i \, e^{\int f(t)dt} = \int g(t) e^{\int f(t)dt} \, dt + C$ — C = arbitrary constant

$$\frac{dP_1}{dt} = -aP_1 \Rightarrow g(t) = 0 \text{ and } f(t) = a \Rightarrow P_1 e^{\int a\,dt} = C_1 \Rightarrow P_1 e^{at} = C_1 \Rightarrow P_1 = C_1 e^{-at}$$

Where C_1 = probability of P_1 at $t = 0$ Assume $C_1 = P_1(0) = 1 \Rightarrow P_1 = e^{-at}$

$$\frac{dP_2}{dt} = aP_1 - bP_2 \Rightarrow g(t) = aP_1 \text{ and } f(t) = b \Rightarrow P_2 e^{bt} = \int aP_1 e^{bt} \, dt + C_2$$

$$= a\int e^{-at} e^{bt} \, dt + C_2 = a\int e^{(b-a)t} \, dt + C_2 = \frac{a}{b-a} e^{(b-a)t} + C_2 \Rightarrow$$

Solution to "Standby" using a Formula cont.

$$P_f = \int_0^t pdf(x)dx$$

$$P_2 = \frac{a}{b-a}e^{-at} + C_2 e^{-bt}$$ Now by assumption $P_2 = 0$ when $t = 0 \Rightarrow C_2 = \frac{-a}{b-a} \Rightarrow$

$$P_2 = \frac{a}{b-a}e^{-at} - \frac{a}{b-a}e^{-bt} = \frac{a}{a-b}e^{-bt} - \frac{a}{a-b}e^{-at}$$

Again since P_1 and P_2 are known, use the fact that $P_1 + P_2 + P_3 = 1$.

$$P_1 + P_2 + P_3 = 1 \;\Rightarrow\; P_3 = 1 - e^{-at} + \frac{a}{a-b}e^{-at} - \frac{a}{a-b}e^{-bt} \;\Rightarrow$$

$$\boxed{P_3 = 1 + \frac{b}{a-b}e^{-at} - \frac{a}{a-b}e^{-bt}}$$

Solution to "Standby" using Convolution

$$P_f = \int_0^t pdf(x)dx$$

A process taken from Calculus called "Convolution" can also be used to calculate P_f of Standby Systems.

Definition:

Let A(t) and B(t) be probabilities of failure of two devices, with device B in Standby of device A, and let a(t) be the derivative of A(t).

The Convolution of A and B = Conv(t) = $\int_0^t B(t-x) \cdot a(x)dx = P_f$

Conv(t) turns out to be the Standby System's Probability of failure P_f.

Solution to "Standby" using Convolution cont.

$$P_f = \int_0^t pdf(x)dx$$

Let $A(x) = 1 - e^{-ax}$, and $B(x) = 1 - e^{-bx}$ be the probabilities of failure of devices A and B. Then $A'(x) = a(x) = ae^{-ax}$, and $B(t-x) = 1 - e^{-b(t-x)}$ since a and b are constant failure rates of devices A and B respectively $\Rightarrow$

$$P_f = Conv(t) = \int_0^t (1 - e^{-b(t-x)}) \cdot ae^{-ax} dx = a\int_0^t (e^{-ax} - e^{-b(t-x)-ax})dx \Rightarrow$$

$$P_f = a\int_0^t (e^{-ax} - e^{-bt-(a-b)x})dx = a\int_0^t e^{-ax}dx - a \cdot e^{-bt}\int_0^t e^{-(a-b)x}dx \Rightarrow$$

$$P_f = (1 - e^{-at}) - \frac{a}{a-b} \cdot e^{-bt}(1 - e^{-(a-b)t}) = 1 - e^{-at} - \frac{a}{a-b}(e^{-bt} - e^{-at}) \Rightarrow$$

$$P_f(sys) = 1 - \frac{a}{a-b}e^{-bt} + \frac{b}{a-b}e^{-at}$$

Solution to "Standby" using Convolution cont.

$$P_f = \int_0^t pdf(x)dx$$

The Convolution Integral can be solved using Laplace and Inverse Laplace Transforms. Simply stated:

$$\text{If } F(t) = \int_0^t B(t-x)\cdot a(x)dx \text{ then } F(t) = L^{-1}\{L[B(t)]\,L[a(t)]\}$$

$$\text{and } L^{-1}\{L[B(t)]\,L[a(t)]\} = L^{-1}\left\{L(1-e^{-bt})\cdot L(ae^{-at})\right\} =$$

$$L^{-1}\left\{\left(\frac{1}{s}-\frac{1}{s+b}\right)\left(\frac{a}{s+a}\right)\right\} = L^{-1}\left\{\frac{a}{s(s+a)}-\frac{a}{(s+a)(s+b)}\right\} =$$

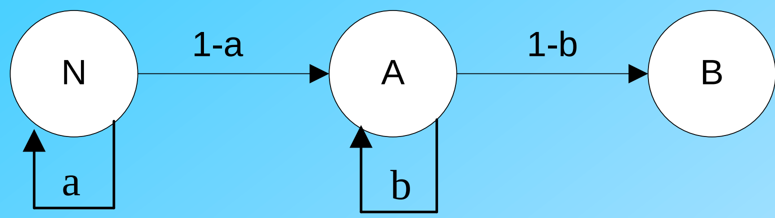
$$1-e^{-at}-\frac{a}{b-a}(e^{-at}-e^{-bt}) \Rightarrow \boxed{F(t) = 1+\frac{a}{b-a}e^{-bt}-\frac{b}{b-a}e^{-at}}$$

Solution to "Standby" using Matrix Algebra

Discrete "Standby" Probability Equation (derived from Transition Matrix)
With respect to Standby TM shown on next slide,
a = probability of transitioning from State N to State N (probability of remaining in State N)
1−a = probability of transitioning from State N to State A
1−b = probability of transitioning from State A to State B
b = probability of transitioning from State A to State A (probability of remaining in State A)
1 = probability of transitioning from State B to State B (probability of remaining in State B)

Let $(N_n \;\; A_n \;\; B_n)$ be a vector representing the probabilities of being in States N, A, and B at time $n\Delta t$. Then the three probabilities at time $(n+1)\Delta t$, can be represented by the vector $(N_{n+1} \;\; A_{n+1} \;\; B_{n+1})$ using simple matrix algebra as shown on next slide.

Solution to "Standby" using Matrix Algebra cont. $P_f = \int_0^t pdf(x)dx$



$$TM = \begin{pmatrix} a & 1-a & 0 \\ 0 & b & 1-b \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} N_{n+1} & A_{n+1} & B_{n+1} \end{pmatrix} = \begin{pmatrix} a & 1-a & 0 \\ 0 & b & 1-b \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} N_n & A_n & B_n \end{pmatrix} = \begin{pmatrix} aN_n & (1-a)N_n + bA_n & (1-b)A_n + B_n \end{pmatrix} \Rightarrow$$

$$N_{n+1} = aN_n, \quad A_{n+1} = (1-a)N_n + bA_n, \quad \text{and} \quad B_{n+1} = (1-b)A_n + B_n$$

$$N_0 = 1 \Rightarrow N_1 = aN_0 = a, \; N_2 = aN_1 \Rightarrow N_2 = aN_1 = a^2 \cdots \Rightarrow$$

$$N_n = a^n$$

$$A_1 = (1-a)N_0 + bA_0 \Rightarrow A_1 = (1-a), \; A_2 = (1-a)N_1 + bA_1 \Rightarrow A_2 = (1-a)\cdot a + b(1-a) = (a+b)\cdot(1-a)$$

$$A_3 = (1-a)N_2 + bA_2 \Rightarrow A_3 = (1-a)a^2 + b(a+b)\cdot(1-a) = (a^2 + ab + b^2)\cdot(1-a)$$

$$A_4 = (1-a)N_3 + bA_3 \Rightarrow A_4 = (1-a)a^3 + b(a^2 + ab + b^2)\cdot(1-a) = (a^3 + a^2 b + ab^2 + b^3)\cdot(1-a) \Rightarrow$$

$$A_n = \frac{(a^n - b^n)}{a-b}\cdot(1-a)$$

$$B_n = 1 - a^n - \frac{a^n - b^n}{a-b}\cdot(1-a) \quad \text{since } 1 = N_n + A_n + B_n$$

Solution to "Standby" using SSD

$$P_f = \int_0^t pdf(x)dx$$

Notes:

State N = Box A operating and Box B in standby state,

State A = Box A Failed and Box B operating state,

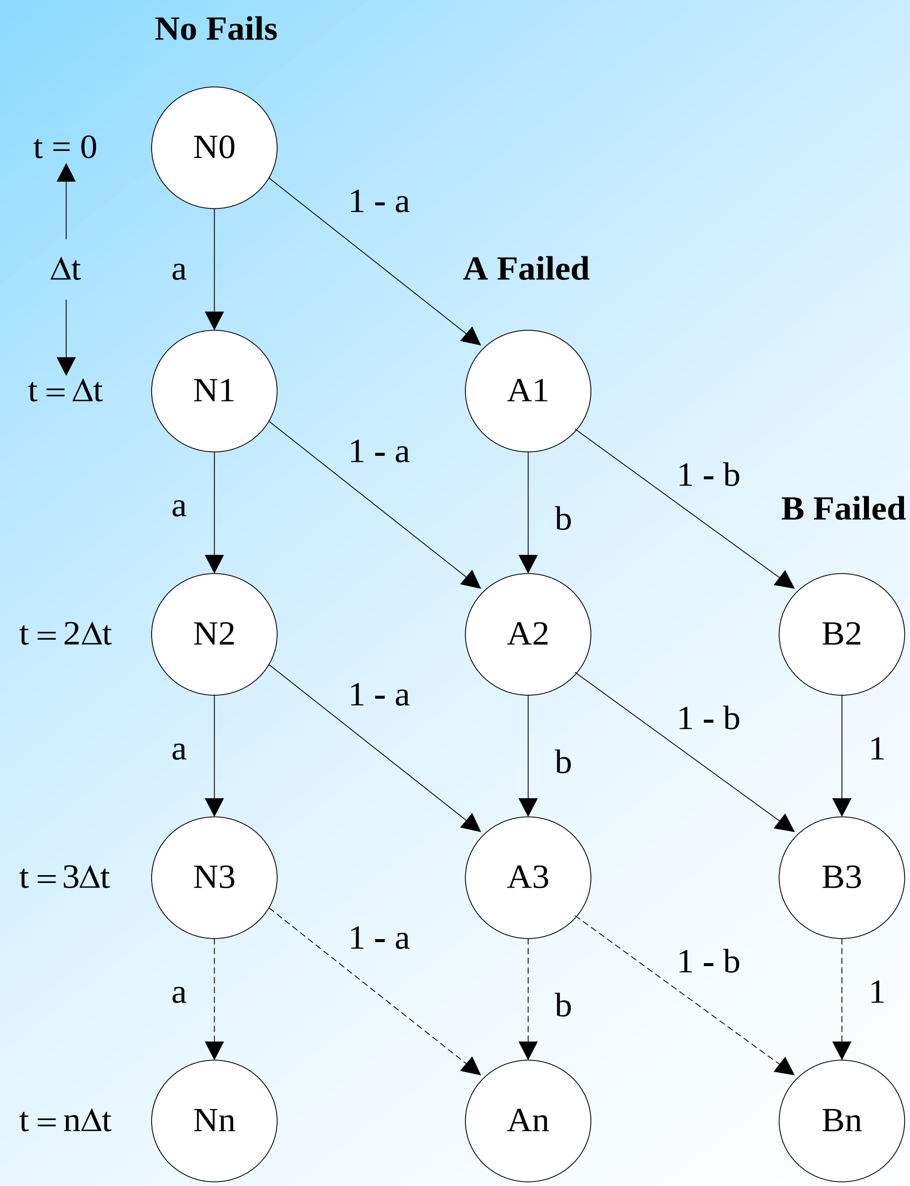
State B = Both Boxes A and B Failed state.

Δt = fixed time interval, n = time increment multiplier,

Nn = Probability of being in State N at time $n\Delta t$ (no failures)

An = Probability of being in State A at time $n\Delta t$ (A failed)

Bn = Probability of being in State B at time $n\Delta t$ (A and B failed)



Solution to "Standby" using SSD cont.

$$P_f = \int_0^t pdf(x)dx$$

Let Ni = Probability of being in State Ni, Ai = Probability of being in State Ai, and Bi = Probability of being in State Bi at time iΔt.

Referring to the above Standby SSD, a = probability of Box A being operational and b = probability of Box B being operational for elapsed time Δt.

$$\text{If } N_0 = 1 \text{ then } N_1 = a$$

$$N_{n+1} = a \cdot N_n \text{ (from SSD)} \Rightarrow N_n = a^n$$

$$A_{n+1} = (1-a) \cdot N_n + b \cdot A_n \Rightarrow$$

$$A_1 = (1-a) \cdot N_0 + b \cdot A_0 = 1-a$$

$$A_2 = (1-a) \cdot N_1 + b \cdot A_1 = a(1-a) + b(1-a) = (a+b)(1-a)$$

$$A_3 = (1-a) \cdot N_2 + b \cdot A_2 = (1-a)a^2 + b(a+b)(1-a) = (a^2 + ab + b^2)(1-a)$$

$$A_4 = (1-a) \cdot N_3 + b \cdot A_3 = (1-a)a^3 + b(a^2 + ab + b^2)(1-a) = (a^3 + a^2 b + ab^2 + b^3)(1-a) \Rightarrow$$

$$\vdots$$

$$A_{n+1} = (1-a)(a^{n+1} - b^{n+1})/(a-b) \text{ or } A_n = (1-a)(a^n - b^n)/(a-b)$$

$$\text{Now } N_n + A_n + B_n = 1 \ \forall n \Rightarrow B_n = 1 - a^n - (1-a)(a^n - b^n)/(a-b) \ \therefore$$

$$N_n = a^n$$

$$A_n = \frac{a^n - b^n}{a-b} \cdot (1-a)$$

$$B_n = 1 - a^n - \frac{a^n - b^n}{a-b} \cdot (1-a) \text{ since } 1 = N_n + A_n + B_n$$

Solution to "Standby" using Arithmetic

$$P_f = \int_0^t pdf(x)dx$$

Another approach uses a spreadsheet and arithmetic derived from the SDE.

$$\frac{dP_1}{dt} = -aP_1, \quad \frac{dP_2}{dt} = aP_1 - bP_2, \quad \frac{dP_3}{dt} = bP_2 \Rightarrow$$

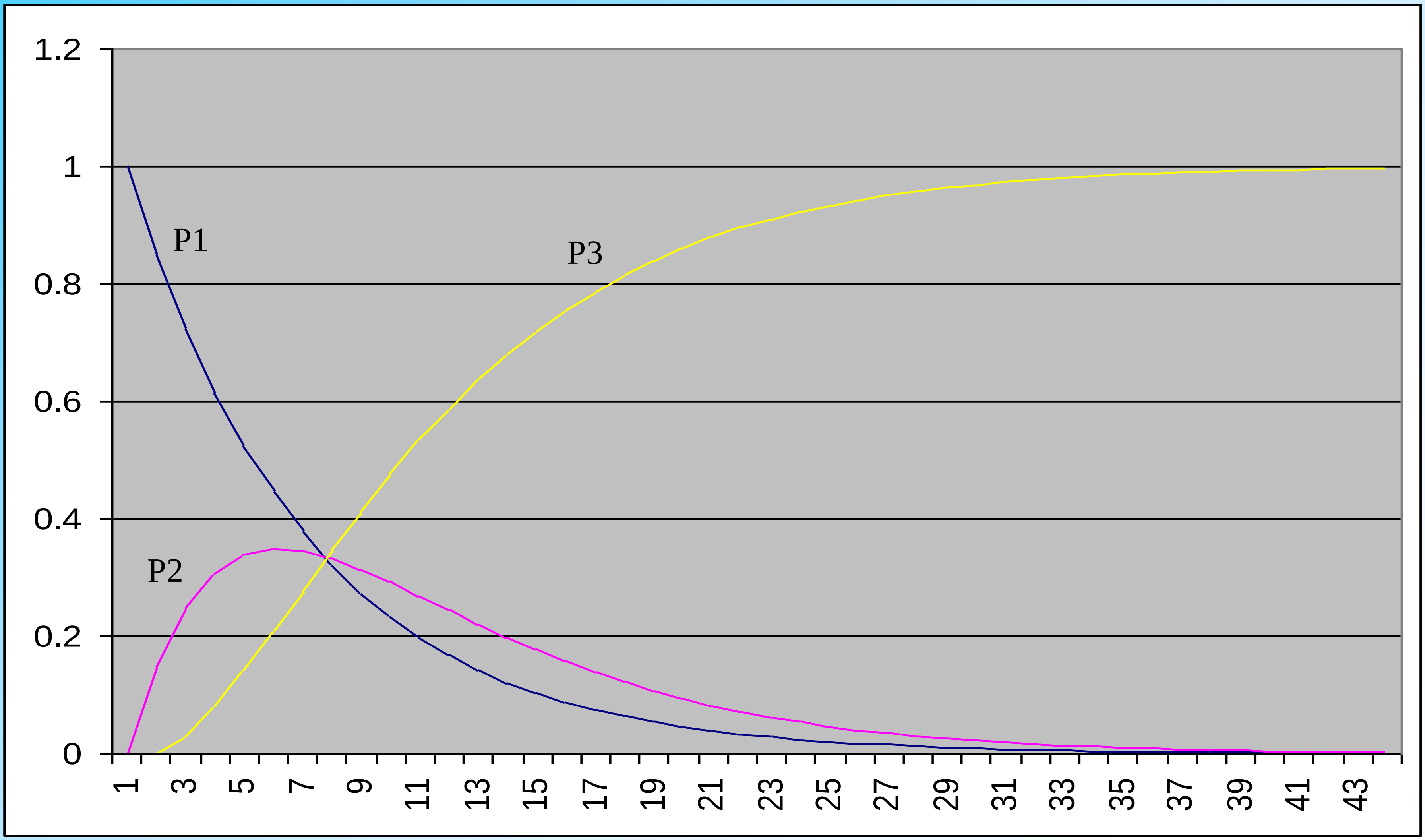
Example: dt = delta t = 0.5, a = 0.3, b = 0.4

	P1- aP1dt	P2+(aP1- bP2)dt	P3+bP2dt
t	P1 = P(State1)	P2 = P(State2)	P3 = P(State3)
0	1	0	0
dt	0.85	0.15	0
2dt	0.7225	0.2475	0.03
3dt	0.614125	0.306375	0.0795
4dt	0.52200625	0.33721875	0.140775
5dt	0.443705313	0.348075938	0.20821875
6dt	0.377149516	0.345016547	0.277833938
7dt	0.320577088	0.332585665	0.346837247
8dt	0.272490525	0.314155095	0.41335438
9dt	0.231616946	0.292197655	0.476185399
10dt	0.196874404	0.268500666	0.53462493
11dt	0.167343244	0.244331693	0.588325063

Solution to "Standby" using Arithmetic cont.

$$P_f = \int_0^t pdf(x)dx$$

Graph output from spreadsheet.



Evidence of SDE Method Limitation

$$P_f = \int_0^t pdf(x)dx$$

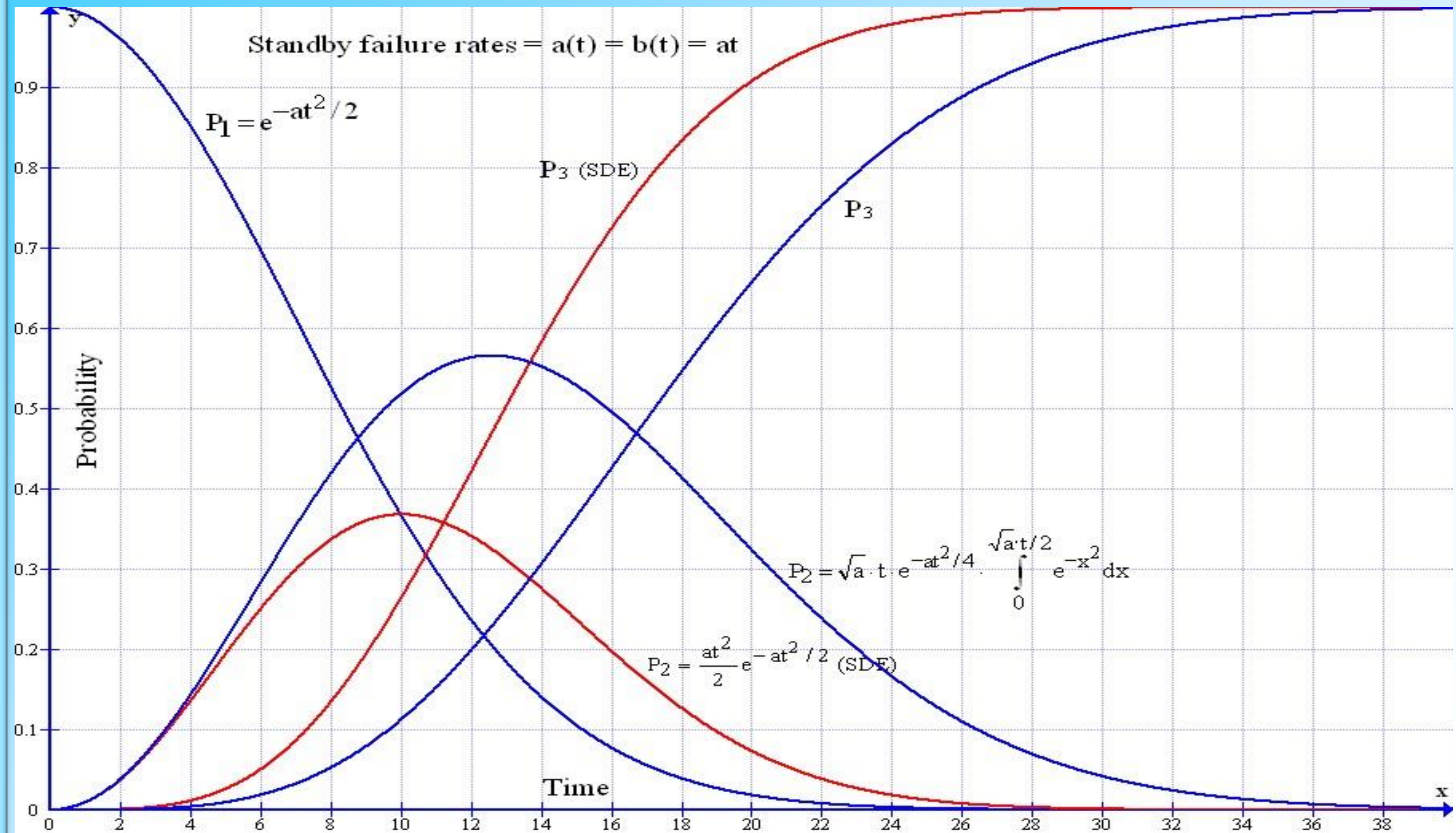
Evidence of a Limitation of Simultaneous Differential Equation Method

Evidence of SDE Method Limitation cont.

$$P_f = \int_0^t pdf(x)dx$$

SDE method is unreliable with respect to <u>non-constant</u> failure rate problems.

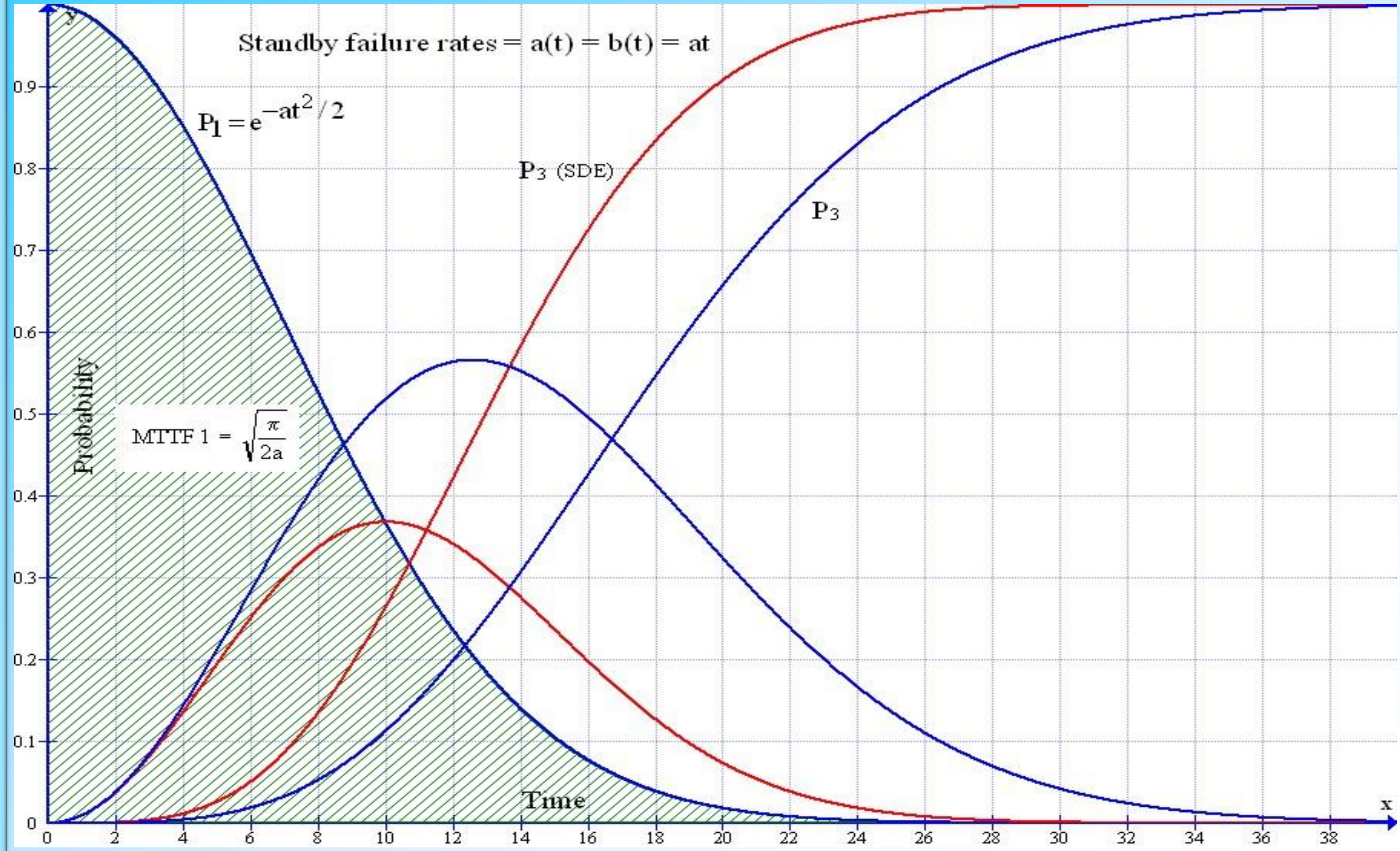
Example: Standby problem with 2 identical devices. (hypothetical FR = a·t)



Evidence of SDE Method Limitation cont.

$$P_f = \int_0^t pdf(x)dx$$

Shaded area under P1 curve represents MTTF of Device 1

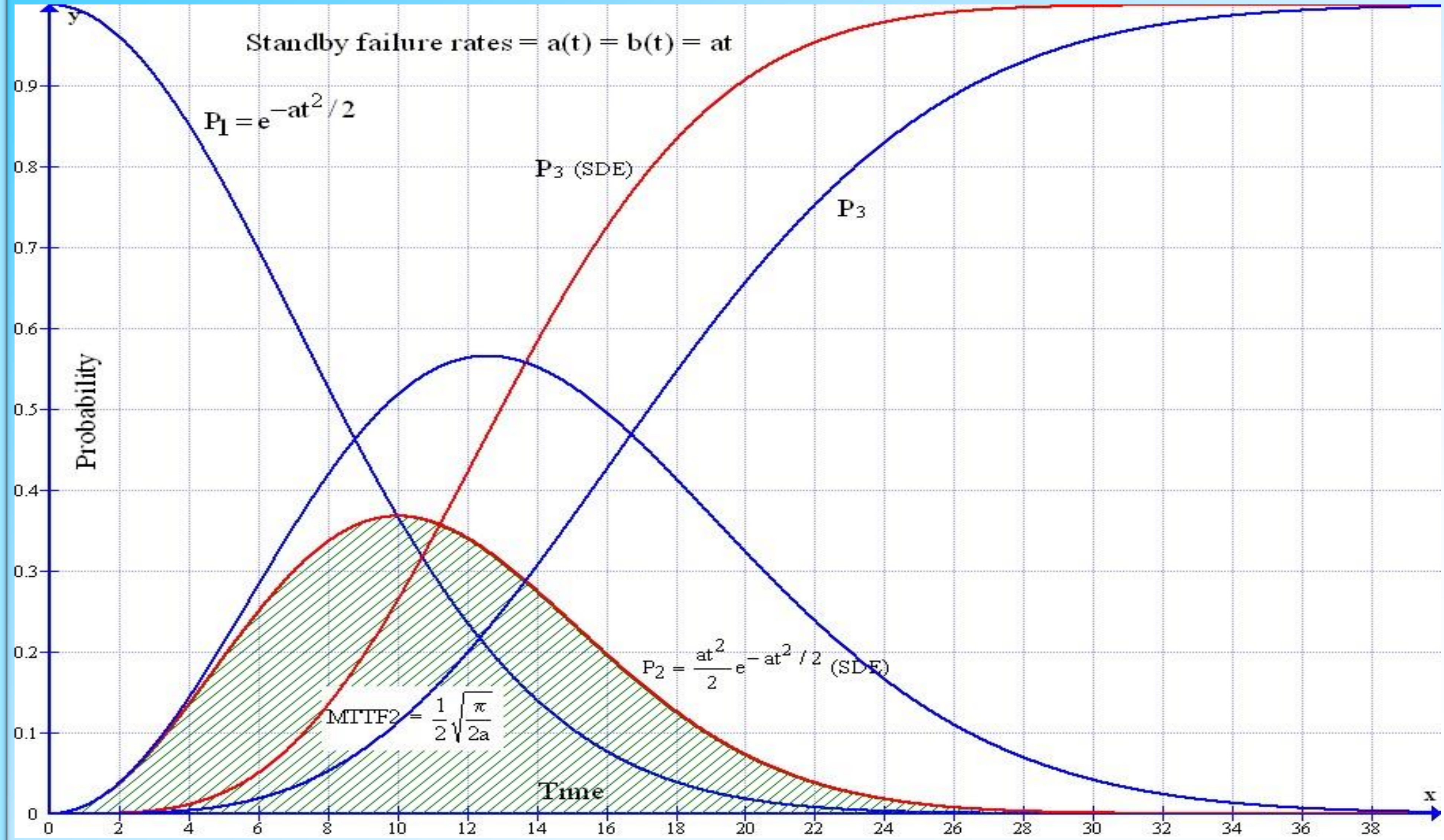


Evidence of SDE Method Limitation cont.

$$P_f = \int_0^t pdf(x)dx$$

Shaded area under P2 curve represents MTTF of Device 2

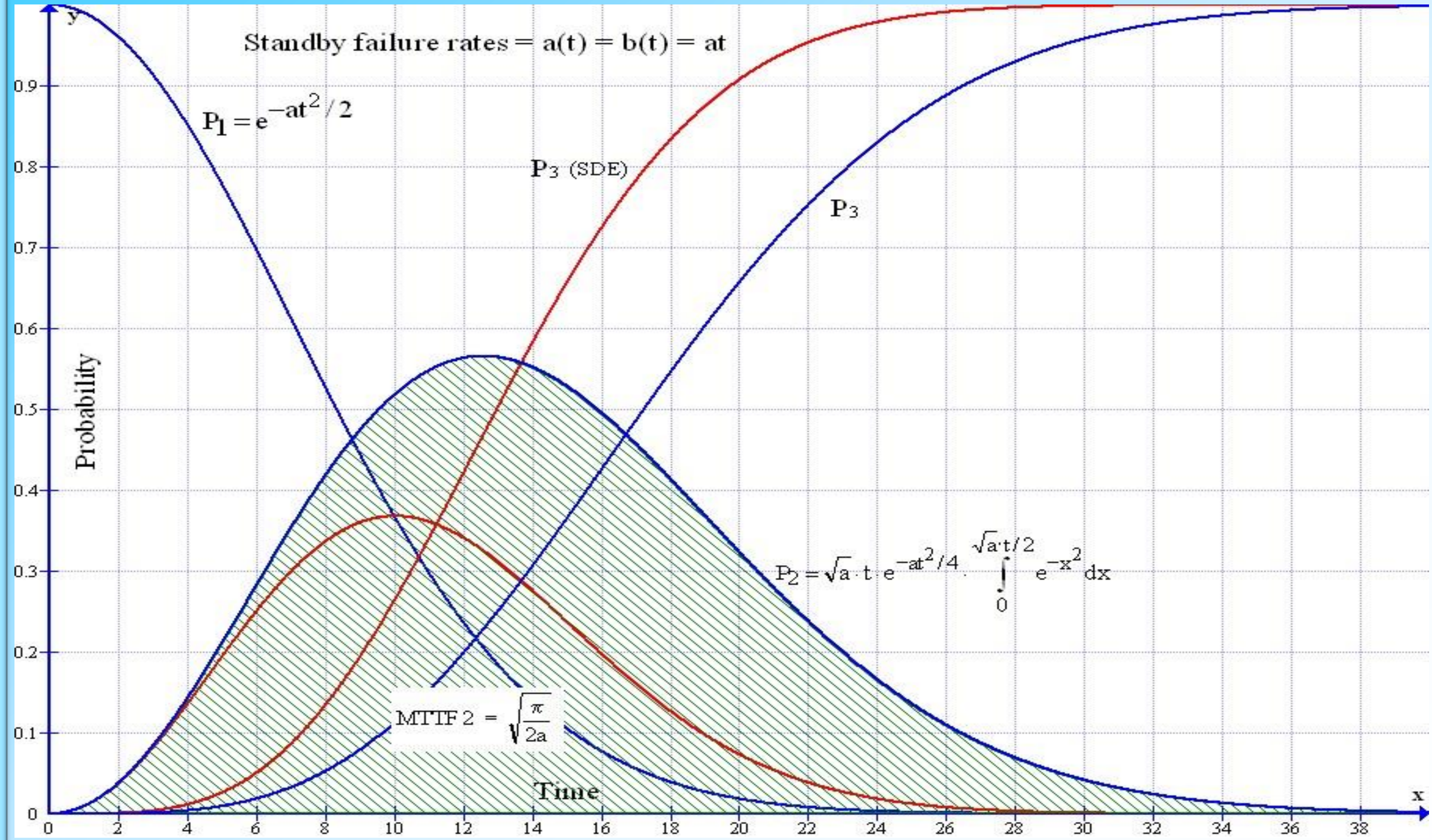
MTTF2 = ½ MTTF1 when calculated using SDE Method (incorrect)



Evidence of SDE Method Limitation cont.

$$P_f = \int_0^t pdf(x)dx$$

Shaded area under P2 curve represents MTTF of Device 2
MTTF2 = MTTF1 using Convolution (correct result)



Summary

$$P_f = \int_0^t pdf(x)dx$$

Summary

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$$P_f = \int_0^t pdf(x)dx$$

Facts to keep in mind:

- FTA exhibits a clear representation of any combinatorial logic process, and capable of handling both constant and non-constant failure rates.
- Integration of non-constant with constant failure rate items may require higher math, but has nothing to do with a system's combinatorial logic.
- MA is a supplement to, and not a replacement for FTA.
- FTA cannot exhibit non-combinatorial logic processes, although FTAs have been used to calculate approximations.
- FTA approximations can be either more or less conservative than exact solution.
- With respect to Reliability, Markov can be thought of as a buzz word for various methodologies used for solving non-combinatorial logic problems.
- Markov techniques can also solve combinatorial problems. (Not recommended)
- State diagrams are used to represent non-combinatorial logic.
- SDE can be very easily determined after the state diagram is constructed.
- Markov requires solutions to a set of n SDE where n = number of states.
- With respect to non-combo problems, Markov adds qualitative and quantitative accuracy over FTA, but requires more work.

Summary cont.

$$P_f = \int_0^t pdf(x)dx$$

Markov Approach (for systems exhibiting non-combinatorial logic):

- Create a system RBD and partition combinatorial and non-combinatorial sections.
  (may be challenging – Differences between Combo and Non-combo problems can be very subtle)
- Use and/or logic (standard FTA methods) for all combinatorial sections.
- Create state diagrams for all non-combinatorial sections.
- Determine set of SDE from state diagrams of each section.
- Find solution set to SDE to determine probability of states.
  (may be challenging – Solutions to SDE is a subject of its own)
- Integrate solutions of all sections and determine probability of all undesirable states (events).

Markov Limitations

- The Markov SDE approach presented in this paper is unreliable when failure rates are non-constant. Math modeling non-combinatorial logic of non-constant failure rate items will require other techniques.
- Markov approach can become difficult when a large number of states are involved.

Where to Get More Information

$$P_f = \int_0^t pdf(x)dx$$

- SAE ARP 4761 Issue 1996-12
- Engineering Reliability Fundamentals & Applications – R. Ramakumar
- System Reliability Theory – A. Hoyland & M. Rausand
- Probabilistic Risk Assessment & Management for Engineers & Scientists – H. Kumamoto & E. Henley
- Modeling for Reliability Analysis – Jan Pukite & Paul Pukite
- Mil-Hdbk 338A – Electronic Reliability Design Handbook
- Mil-Std 756B – Reliability Modeling and Prediction

Contact Information

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Questions

$$P_f = \int_0^t pdf(x)dx$$

Thank you for your attention.

Do you have any questions?